On the reduction of Alperin's Conjecture to the quasi-simple groups

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1. Introduction

1.1. In [10, Chap. 16] we propose a refinement of Alperin's Conjecture whose possible proof can be reduced to check that this refinement holds on the so-called *quasi-simple* groups. To carry out this checking obviously depends on admitting the *Classification of the Finite Simple Groups*, and our proof of the reduction itself uses the *solvability* of the *outer automorphism group* of a finite simple group, a known fact whose actual proof depends on this classification [10, 16.11].

1.2. Unfortunately, on the one hand in [10, Chap. 16] our proof also depends on checking, in the list of *quasi-simple* groups, the technical condition [10, 16.22.1] — which, as a matter of fact, is not always fulfilled: it is not difficult to exhibit a counter-example from Example 4.2 in [12]! – and on the other hand, in the second half of [10, Chap. 16] some arguments have been scratched. The purpose of this paper is to remove that troublesome condition and to repair the bad arguments there.† Eventually, we find the better result stated below.

1.3. Let us be more explicit. Let p be a prime number, k an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring of characteristic zero admitting k as the *residue* field, $\hat{G}$ a k^*-group of finite k^*-quotient G [10, 1.23], b a block of $\hat{G}$ [10, 1.25] and $\mathcal{G}_k(\hat{G}, b)$ the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finite generated $k_*\hat{G}b$-modules [10, 14.3]. In [10, Chap. 14], choosing a maximal Brauer $(b, \hat{G})$-pair (P, e), the existence of a suitable k^*-$\mathfrak{Gr}$-valued functor $\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}$ over some full subcategory $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}$ of the *Frobenius P-category* $\mathcal{F}_{(b,\hat{G})}$ [10, 3.2] allows us to consider an inverse limit of Grothendieck groups — noted $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$ and called the *Grothendieck group of* $\mathcal{F}_{(b,\hat{G})}$ — such that Alperin's Conjecture is actually equivalent to the existence of an $\mathcal{O}$-module isomorphism [10, I32 and Corollary 14.32]

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad \qquad 1.3.1.$$

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† In particular, this paper has to be considered as an ERRATUM of a partial contents of [10, Chap. 16] going from [10, 16.20] to the end of the chapter.

1.4. Denote by $\mathrm{Out}_{k^*}(\hat{G})$ the group of *outer k^*-automorphisms* of $\hat{G}$ and by $\mathrm{Out}_{k^*}(\hat{G})_b$ the stabilizer of b in $\mathrm{Out}_{k^*}(\hat{G})$; it is clear that $\mathrm{Out}_{k^*}(\hat{G})_b$ acts on $\mathcal{G}_k(\hat{G}, b)$, and in [10, 16.3 and 16.4] we show that this group still acts on $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathrm{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$. Our purpose is to show that the following statement

(Q)   *For any k^*-group with finite k^*-quotient G and any block b of $\hat{G}$, there is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism*

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathrm{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad\qquad 1.4.1$$

can be proved by checking that it holds in all the cases where G contains a normal noncommutative simple subgroup S such that $C_G(S) = \{1\}$, p divides $|S|$ and G/S is a cyclic p'-group.

1.5. To carry out this purpose, in [10, Chap. 15] we develop reduction results relating both members of isomorphism 1.4.1 with the Grothendieck groups coming from suitable proper *normal sub-blocks*; recall that a *normal sub-block* of $(b, \hat{G})$ is a pair $(c, \hat{H})$ formed by a normal k^*-subgroup $\hat{H}$ of $\hat{G}$ and a block c of $\hat{H}$ fulfilling $cb \neq 0$. It is one of this reduction results — namely in the case where $\hat{G}/\hat{H}$ is a p'-group [10, Proposition 15.19] — that we improve here, allowing us to remove condition [10, 16.22.1]. Then, following the same strategy as in [10, Chap. 16], we will show that in [10, Chap. 16] all the statements including this condition in their hypothesis can be replaced by stronger results and, at the end, we succeed in replacing [10, Theorem 16.45] by the following more precise result.

Theorem 1.6. *Assume that any block $(c, \hat{H})$ having a normal sub-block $(d, \hat{S})$ of positive defect such that the k^*-quotient S of $\hat{S}$ is simple, H/S is a cyclic p'-group and $C_H(S) = \{1\}$, fulfills the following two conditions*

1.6.1 $\mathrm{Out}(S)$ *is solvable.*

1.6.2   *There is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{H})_c$-module isomorphism*

$$\mathcal{G}_k(\hat{H}, c) \cong \mathcal{G}_k(\mathcal{F}_{(c,\hat{H})}, \widehat{\mathrm{aut}}_{(\mathcal{F}_{(c,\hat{H})})^{\mathrm{nc}}}).$$

Then, for any block $(b, \hat{G})$ there is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathrm{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}}) \qquad\qquad 1.6.3.$$

2. Notation and quoted results

2.1. We already have fixed p, k and $\mathcal{O}$. We only consider k-algebras A of finite dimension and denote by $J(A)$ the *radical* and by A^* the group of invertible elements of A. Let G be a finite group; a *G-algebra* is a k-algebra A endowed with a G-action [4] and we denote by A^G the subalgebra of G-fixed elements. A G-algebra homomorphism from A to another G-algebra A' is a *not necessarily unitary* k-algebra homomorphism $f : A \to A'$ compatible with the G-actions; we say that f is an *embedding* whenever

$$\mathrm{Ker}(f) = \{0\} \quad \text{and} \quad \mathrm{Im}(f) = f(1_A)A'f(1_A) \qquad 2.1.1.$$

2.2. Recall that, for any subgroup H of G, a *point* α of H on A is an $(A^H)^*$-conjugacy class of primitive idempotents of A^H and the pair H_α is a *pointed group* on A [5, 1.1]. For any $i \in \alpha$, iAi has an evident structure of H-algebra and we denote by A_α one of these mutually $(A^H)^*$-conjugate H-algebras and by $A(H_\alpha)$ the *simple quotient* of A^H determined by α. A second pointed group K_β on A is *contained* in H_α if $K \subset H$ and, for any $i \in \alpha$, there is $j \in \beta$ such that [5, 1.1]

$$ij = j = ji \qquad 2.2.2;$$

then, it is clear that the $(A^K)^*$-conjugation induces K-algebra embeddings

$$f_\beta^\alpha : A_\beta \longrightarrow \mathrm{Res}_K^H(A_\alpha) \qquad 2.2.3.$$

2.3. For any p-subgroup P of G we consider the *Brauer quotient* and the *Brauer homomorphism* [1, 1.2]

$$\mathrm{Br}_P^A : A^P \longrightarrow A(P) = A^P \Big/ \sum_Q A_Q^P \qquad 2.3.1,$$

where Q runs over the set of proper subgroups of P, and call *local* any point γ of P on A not contained in $\mathrm{Ker}(\mathrm{Br}_P^A)$ [5, 1.1]. Recall that *a local pointed group P_γ contained in H_α is maximal if and only if* $\mathrm{Br}_P(\alpha) \subset A(P_\gamma)_P^{N_H(P_\gamma)}$ [5, Proposition 1.3] and then *the P-algebra A_γ — called a source algebra of A_α — is Morita equivalent to A_α* [9, 6.10]; moreover, *the maximal local pointed groups P_γ contained in H_α — called the defect pointed groups of H_α — are mutually H-conjugate* [5, Theorem 1.2].

2.4. Let us say that A is a *p-permutation G-algebra* if a Sylow p-subgroup of G stabilizes a basis of A [1, 1.1]. In this case, recall that if P is a p-subgroup of G and Q a normal subgroup of P then the corresponding Brauer homomorphisms induce a k-algebra isomorphism [1, Proposition 1.5]

$$\big(A(Q)\big)(P/Q) \cong A(P) \qquad 2.4.1;$$

moreover, choosing a point α of G on A, we call *Brauer (α, G)-pair* any pair (P, e_A) formed by a p-subgroup P of G such that $\mathrm{Br}_P^A(\alpha) \neq \{0\}$ and by a primitive idempotent e_A of the center $Z\big(A(P)\big)$ of $A(P)$ such that

$$e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\} \qquad\qquad 2.4.2;$$

note that any local pointed group Q_δ on A *contained* in G_α determines a Brauer (α, G)-pair (Q, f_A) fulfilling $f_A \cdot \mathrm{Br}_Q^A(\delta) \neq \{0\}$.

2.5. It follows from [1, Theorem 1.8] that *the inclusion between the local pointed groups on A induces an inclusion between the Brauer (α, G)-pairs*; explicitly, if (P, e_A) and (Q, f_A) are two Brauer (α, G)-pairs then we have

$$(Q, f_A) \subset (P, e_A) \qquad\qquad 2.5.1$$

whenever there are local pointed groups P_γ and Q_δ on A fulfilling

$$Q_\delta \subset P_\gamma \subset G_\alpha \quad , \quad f_A \cdot \mathrm{Br}_Q^A(\delta) \neq \{0\} \quad \text{and} \quad e_A \cdot \mathrm{Br}_P^A(\gamma) \neq \{0\} \qquad 2.5.2.$$

Actually, according to the same result, for any p-subgroup P of G, any primitive idempotent e_A of $Z\big(A(P)\big)$ fulfilling $e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\}$ and any subgroup Q of P, there is a unique primitive idempotent f_A of $Z\big(A(Q)\big)$ fulfilling

$$e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\} \quad \text{and} \quad (Q, f_A) \subset (P, e_A) \qquad\qquad 2.5.3.$$

Once again, *the maximal Brauer (α, G)-pairs are pairwise G-conjugate* [1, Theorem 1.14].

2.6. For inductive purposes, we have to consider a k^*-group $\hat{G}$ of finite k^*-quotient G [10, 1.23] rather than a finite group; moreover, we are specially interested in the G-algebras A endowed with a k^*-group homomorphism $\rho : \hat{G} \to A^*$ inducing the action of G on A, called *$\hat{G}$-interior algebras*; in this case, for any pointed group H_α — also noted $\hat{H}_\alpha$ — on A, $A_\alpha = iAi$ has a structure of *$\hat{H}$-interior algebra* mapping $\hat{y} \in \hat{H}$ on $\rho(\hat{y})i = i\rho(\hat{y})$; moreover, setting $\hat{x} \cdot a \cdot \hat{y} = \rho(\hat{x})a\rho(\hat{y})$ for any $a \in A$ and any $\hat{x}, \hat{y} \in \hat{G}$, a $\hat{G}$-interior algebra homomorphism from A to another $\hat{G}$-interior algebra A' is a G-algebra homomorphism $f : A \to A'$ fulfilling

$$f(\hat{x} \cdot a \cdot \hat{y}) = \hat{x} \cdot f(a) \cdot \hat{y} \qquad\qquad 2.6.1.$$

2.7. In particular, if H_α and K_β are two pointed groups on A, we say that an injective group homomorphism $\varphi : K \to H$ is an *A-fusion from K_β to H_α* whenever there is a K-interior algebra *embedding*

$$f_\varphi : A_\beta \longrightarrow \mathrm{Res}_K^H(A_\alpha) \qquad\qquad 2.7.1$$

such that the inclusion $A_\beta \subset A$ and the composition of f_φ with the inclusion $A_\alpha \subset A$ are A^*-conjugate; we denote by $F_A(K_\beta, H_\alpha)$ the set of H-conjugacy classes of A-fusions from K_β to H_α and, as usual, we write $F_A(H_\alpha)$ instead of $F_A(H_\alpha, H_\alpha)$. If $A_\alpha = iAi$ for $i \in \alpha$, it follows from [6, Corollary 2.13] that we have a group homomorphism

$$F_A(H_\alpha) \longrightarrow N_{A_\alpha^*}(H \cdot i)/H \cdot (A_\alpha^H)^* \qquad 2.7.2$$

and if H is a p-group then we consider the k^*-group $\hat{F}_A(H_\alpha)$ defined by the *pull-back*

$$\begin{array}{ccc} F_A(H_\alpha) & \longrightarrow & N_{A_\alpha^*}(H \cdot i)/H \cdot (A_\alpha^H)^* \\ \uparrow & & \uparrow \\ \hat{F}_A(H_\alpha) & \longrightarrow & N_{A_\alpha^*}(H \cdot i)/H \cdot (i + J(A_\alpha^H)) \end{array} \qquad 2.7.3.$$

2.8. We also consider the *mixed* situation of an $\hat{H}$-*interior G-algebra B* where $\hat{H}$ is a k^*-subgroup of $\hat{G}$ and B is a G-algebra endowed with a *compatible* $\hat{H}$-interior algebra structure, in such a way that the $k_*\hat{G}$-module $B \otimes_{k_*\hat{H}} k_*\hat{G}$ endowed with the product

$$(a \otimes \hat{x}).(b \otimes \hat{y}) = ab^{x^{-1}} \otimes \hat{x}\hat{y} \qquad 2.8.1,$$

for any $a, b \in B$ and any $\hat{x}, \hat{y} \in \hat{G}$, and with the group homomorphism mapping $\hat{x} \in \hat{G}$ on $1_B \otimes \hat{x}$, becomes a $\hat{G}$-interior algebra — simply noted $B \otimes_{\hat{H}} \hat{G}$. For instance, for any p-subgroup P of $\hat{G}$, $A(P)$ is a $C_{\hat{G}}(P)$-interior $N_G(P)$-algebra.

2.9. Obviously, the *group algebra* $k_*\hat{G}$ is a p-permutation $\hat{G}$-interior algebra and, for any block b of $\hat{G}$, the $((k_*\hat{G})^G)^*$-conjugacy class $\alpha = \{b\}$ is a *point* of G on $k_*\hat{G}$. Moreover, for any p-subgroup P of $\hat{G}$, the Brauer homomorphism $\mathrm{Br}_P = \mathrm{Br}_P^{k_*\hat{G}}$ induces a k-algebra isomorphism [8, 2.8.4]

$$k_*C_{\hat{G}}(P) \cong (k_*\hat{G})(P) \qquad 2.9.1;$$

thus, up to identification throughout this isomorphism, in a Brauer (α, G)-pair (P, e) as defined above — called *Brauer $(b, \hat{G})$-pair* from now on — e is nothing but a block of $C_{\hat{G}}(P)$ such that $e\mathrm{Br}_P(b) \neq 0$. It is handy to consider the quotient

$$\bar{C}_{\hat{G}}(P) = C_{\hat{G}}(P)/Z(P) \qquad 2.9.2$$

and we denote by

$$\bar{\mathrm{Br}}_P : (k_*\hat{G})^Q \longrightarrow k_*\bar{C}_{\hat{G}}(P) \qquad 2.9.3$$

the corresponding homomorphism; recall that the image $\bar{e}$ of e in $k_*\bar{C}_{\hat{G}}(P)$ is a block of $\bar{C}_{\hat{G}}(P)$ and that the *Brauer First Main Theorem* affirms that

(P, e) *is maximal if and only if the k-algebra $k_* \bar{C}_{\hat{G}}(P) \bar{e}$ is simple and the inertial quotient*

$$E_G(P, e) = N_{\hat{G}}(P, e)/P \cdot C_{\hat{G}}(P) \qquad 2.9.4$$

is a p'-group [9, Theorem 10.14].

2.10. In this case, the *Frobenius P-category* $\mathcal{F} = \mathcal{F}_{(b, \hat{G})}$ of b [10, 3.2] is, up to equivalence, the category where the objects are the Brauer $(b, \hat{G})$-pairs (Q, f) and the morphisms are determined by the homomorphisms between the corresponding p-groups induced by the *inclusion* between Brauer $(b, \hat{G})$-pairs and by the G-conjugation. Then, we say that (Q, f) is *nilcentralized* if the block f of $C_{\hat{G}}(Q)$ is *nilpotent* [10, Proposition 7.2], we denote by $\mathcal{F}^{\mathrm{nc}}$ the full subcategory of $\mathcal{F}$ over the set of nilcentralized Brauer $(b, \hat{G})$-pairs, and consider the *proper category of $\mathcal{F}^{\mathrm{nc}}$-chains* $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}})$ [10, A2.8] and the *automorphism functor* [10, Proposition A2.10]

$$\mathfrak{aut}_{\mathcal{F}^{\mathrm{nc}}} : \mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}}) \longrightarrow \mathfrak{Gr} \qquad 2.10.1,$$

where $\mathfrak{Gr}$ denotes the category of finite groups, mapping any $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}})$-object $(\mathfrak{q}, \Delta_n)$ — $\mathfrak{q}$ being a functor from the *ordered n-simplex* Δ_n to $\mathcal{F}^{\mathrm{nc}}$ — to its $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}})$-automorphism group — the automorphism group of the functor $\mathfrak{q}$, simply noted $\mathcal{F}(\mathfrak{q})$.

2.11. If (Q, f) is a nilcentralized Brauer $(b, \hat{G})$-pair, f determines a unique *local point* δ of Q on $k_* \hat{G}$ since $k_* C_{\hat{G}}(Q) f$ has a unique isomorphism class of simple modules [7, (1.9.1)]; now, the action of $N_G(Q, f)$ on the simple k-algebra $(k_* \hat{G})(Q_\delta)$ determines a k^*-group $\hat{N}_G(Q, f)$ and it is clear that the the corresponding k^*-subgroup $\hat{C}_G(Q)$ is canonically isomorphic to $C_{\hat{G}}(Q)$, so that the "difference" $\hat{N}_G(Q, f) * N_{\hat{G}}(Q, f)^\circ$ [8, 5.9] admits a normal subgroup isomorphic to $C_G(Q)$; then, up to identification, we define

$$\hat{E}_G(Q, f) = \big(\hat{N}_G(Q, f) * N_{\hat{G}}(Q, f)^\circ\big)/Q \cdot C_G(Q) \qquad 2.11.1;$$

note that from [6, Theorem 3.1] and [8, Proposition 6.12], suitable extended to k^*-groups, we obtain *canonical k^*-group isomorphisms* (cf. 2.8.3)

$$\hat{E}_G(Q, f)^\circ \cong \hat{F}_{k_* \hat{G}}(Q_\delta) \cong \hat{F}_{(k_* \hat{G})_\delta}(Q_\delta) \qquad 2.11.2.$$

2.12. In [10, Theorem 11.32] we prove that the functor $\mathfrak{aut}_{\mathcal{F}^{\mathrm{nc}}}$ above can be lifted to a functor

$$\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}} : \mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad 2.12.1,$$

where k^*-$\mathfrak{Gr}$ denotes the category of k^*-groups with finite k^*-quotient, mapping any $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{nc}})$-object $(\mathfrak{q}, \Delta_n)$ on the corresponding k^*-subgroup $\hat{\mathcal{F}}(\mathfrak{q})$

of $\hat{E}_G\big(\mathfrak{q}(n)\big)$. Then, denoting by $\mathfrak{g}_k : k^*\text{-}\mathfrak{Gr} \to \mathcal{O}\text{-}\mathfrak{mod}$ the functor determined by the *Grothendieck groups* and the *restriction maps*, we define [10, 14.3.3]

$$\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{nc}}) = \varprojlim (\mathfrak{g}_k \circ \widehat{\mathfrak{aut}}_{\mathcal{F}^{nc}}) \qquad 2.12.2$$

More precisely, we say that a Brauer $(b, \hat{G})$-pair is *$\mathcal{F}$-selfcentralizing* if the block $\bar{f}$ of $\bar{C}_{\hat{G}}(Q)$ has *defect* zero [10, Corollary 7.3]; denoting by $\mathcal{F}^{sc}$ the full subcategory of $\mathcal{F}$ over the set of selfcentralizing Brauer $(b, \hat{G})$-pairs and by $\widehat{\mathfrak{aut}}_{\mathcal{F}^{sc}}$ the corresponding restriction, in [10, Corollary 14.7] we prove that

$$\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{nc}}) \cong \varprojlim (\mathfrak{g}_k \circ \widehat{\mathfrak{aut}}_{\mathcal{F}^{sc}}) \qquad 2.12.3.$$

2.13. On the other hand, for any p-subgroup Q of $\hat{G}$ and any k^*-subgroup $\hat{H}$ of $N_{\hat{G}}(Q)$ containing $Q \cdot C_{\hat{G}}(Q)$, we have

$$\mathrm{Br}_Q\big((k_*\hat{G})^H\big) = (k_*\hat{G})(Q)^H \qquad 2.13.1$$

and therefore *any block f of $C_{\hat{G}}(Q)$ determines a unique point β of H on $k_*\hat{G}$ such that H_β contains Q_δ for a local point δ of Q on k_*G fulfilling* [7, Lemma 3.9]

$$f \cdot \mathrm{Br}_Q(\delta) \neq \{0\} \qquad 2.13.2.$$

Recall that, if R is a subgroup of Q such that $C_{\hat{G}}(R) \subset \hat{H}$ then the blocks of $C_{\hat{G}}(R) = C_{\hat{H}}(R)$ determined by (Q, f) from $\hat{G}$ and from $\hat{H}$ coincide [1, Theorem 1.8].

2.14. Moreover, denote by γ the local point of P on $k_*\hat{G}$ determined by e and set $E_G(P, e) = E_G(P_\gamma)$; since $E_G(P_\gamma)$ is a p'-group, it follows from [9, Lemma 14.10] that the short exact sequence

$$1 \longrightarrow P/Z(P) \longrightarrow N_G(P_\gamma)/C_G(P) \longrightarrow E_G(P_\gamma) \longrightarrow 1 \qquad 2.14.1$$

is split and that all the splittings are conjugate to each other; thus, any splitting determines an action of $E_G(P_\gamma)$ on P and it is easily checked that the corresponding semidirect product

$$\hat{L}_G(P_\gamma) = P \rtimes \hat{E}_G(P_\gamma)^\circ \qquad 2.14.2$$

does not depend on our choice. Then, it follows from [9, Theorem 12.8] that we have a unique $\big((k_*\hat{G})_\gamma^P\big)^*$-conjugacy class of unitary P-interior algebra homomorphisms

$$l_\gamma : k_*\hat{L}_G(P_\gamma) \longrightarrow (k_*\hat{G})_\gamma \qquad 2.14.3$$

which are also $k(P \times P)$-module *direct injections*.

3. Normal sub-blocks

3.1. Let $\hat{G}$ be a k^*-group with finite k^*-quotient G, $\hat{H}$ a normal k^*-subgroup of $\hat{G}$, b a block of $\hat{G}$ and c a block of $\hat{H}$ fulfilling $cb \neq 0$. Note that we have $b\mathrm{Tr}_{\hat{G}_c}^{\hat{G}}(c) = b$ where $\hat{G}_c$ denotes the stabilizer of c in $\hat{G}$; thus, considering the $\hat{G}$-stable semisimple k-subalgebra $\sum_{\hat{x}} k \cdot bc^{\hat{x}}$ of $k_*\hat{G}$, where $\hat{x} \in \hat{G}$ runs over a set of representatives for $\hat{G}/\hat{G}_c$, it follows from [11, Proposition 3.5] that bc is a block of $\hat{G}_c$ and then from [11, Proposition 3.2] that we have

$$k_*\hat{G}b \cong \mathrm{Ind}_{\hat{G}_c}^{\hat{G}}(k_*\hat{G}_c\, bc) \qquad\qquad 3.1.1,$$

so that the source algebras of the block b of $\hat{G}$ and of the block bc of $\hat{G}_c$ are isomorphic.

3.2. Thus, from now on we assume that $\hat{G}$ fixes c, so that we have $bc = b$ and, in particular, $\alpha = \{c\}$ is a point of $\hat{G}$ on $k_*\hat{H}$ (cf. 2.2). Let (Q, f) be a maximal Brauer $(c, \hat{H})$-pair and denote by $N_{\hat{G}}(Q, f)$ the stabilizer of (Q, f) in $\hat{G}$, setting

$$C_{\hat{G}}(Q, f) = C_{\hat{G}}(Q) \cap N_{\hat{G}}(Q, f) \qquad\qquad 3.2.1;$$

by the *Frattini argument*, we clearly get

$$\hat{G} = \hat{H} \cdot N_{\hat{G}}(Q, f) \qquad\qquad 3.2.2;$$

as in 2.11 above, $N_G(Q, f)$ acts on the simple k-algebra $k_*\bar{C}_{\hat{H}}(Q)\bar{f}$ (cf. 2.9), so that we get a k^*-group $\hat{N}_G(Q, f)$ and the "difference" $N_{\hat{G}}(Q, f) * \hat{N}_G(Q, f)^\circ$ contains a normal subgroup canonically isomorphic to $C_H(Q)$; note that we have (cf. 2.11.1)

$$\hat{E}_H(Q, f) \subset \big(N_{\hat{G}}(Q, f) * \hat{N}_G(Q, f)^\circ\big)/Q \cdot C_H(Q) \qquad\qquad 3.2.3.$$

Moreover, $C_G(Q, f)$ acts on the k^*-group $\hat{E}_H(Q, f)$ acting trivially on the k^*-quotient $E_H(Q, f)$, and therefore, denoting by $S_{\hat{G}}(Q, f)$ the kernel of this action, the quotient

$$Z = C_{\hat{G}}(Q, f)/S_{\hat{G}}(Q, f) \qquad\qquad 3.2.4$$

is an Abelian p'-group.

3.3. More precisely, denoting by δ the local point of Q on $k_*\hat{H}$ determined by f (cf. 2.11) and choosing $j \in \delta$, for any $\hat{x} \in N_{\hat{G}}(Q, f)$ there is $a_{\hat{x}} \in \big((k_*\hat{H})^Q\big)^*$ such that $j^{\hat{x}} = j^{a_{\hat{x}}}$ and, in particular, the element $\hat{x}(a_{\hat{x}})^{-1}$ centralizes j; hence, choosing a set of representatives $\mathcal{X} \subset N_{\hat{G}}(Q, f)$ for the quotient $N_{\hat{G}}(Q, f)/N_{\hat{H}}(Q, f)$, the element $\hat{x}(a_{\hat{x}})^{-1}$ normalizes the *source algebra* $B = j(k_*\hat{H})j$ of c for any $\hat{x} \in \mathcal{X}$, and we easily get

$$D = j(k_*\hat{G})j = \bigoplus_{\hat{x} \in \mathcal{X}} \hat{x}(a_{\hat{x}})^{-1} \cdot B \qquad\qquad 3.3.1.$$

It is clear that, for any $\hat{x} \in C_{\hat{G}}(Q, f)$, the element $\hat{x}(a_{\hat{x}})^{-1}$ induces a Q-in-terior algebra automorphism of B and it follows from [8, Proposition 14.9] that if $\hat{x}$ belongs to $S_{\hat{G}}(Q, f)$ then the element $\hat{x}(a_{\hat{x}})^{-1}$ induces an *interior* automorphism of the Q-interior algebra B; thus, up to modifying our choice of $a_{\hat{x}}$, we may assume that $\hat{x}(a_{\hat{x}})^{-1}$ centralizes B. From now on, we assume that $\hat{x}(a_{\hat{x}})^{-1}$ centralizes B for any $\hat{x} \in S_{\hat{G}}(Q, f)$, so that Z acts on B, and that the elements of $\mathcal{X}$ belonging to $C_{\hat{G}}(Q, f) \cdot N_{\hat{H}}(Q, f)$ are actually chosen in $C_{\hat{G}}(Q, f)$.

3.4. On the other hand, denote by $\hat{C}_G(Q, f)$ the corresponding k^*-subgroup of $\hat{N}_G(Q, f)$ and set

$$\hat{C}_H^G(Q, f) = \big(C_{\hat{G}}(Q, f) * \hat{C}_G(Q, f)^{\circ}\big)/C_H(Q) \qquad 3.4.1;$$

then, since $\bar{f}$ is a block of defect zero of $\bar{C}_{\hat{H}}(Q)$, we have [11, Theorem 3.7]

$$k_* \bar{C}_{\hat{G}}(Q, f)\bar{f} \cong k_* \bar{C}_{\hat{H}}(Q)\bar{f} \otimes_k k_* \hat{C}_H^G(Q, f) \qquad 3.4.2;$$

more generally, we still have

$$k_* \bar{C}_{\hat{G}}(Q) \mathrm{Tr}_{\bar{C}_{\hat{G}}(Q, f)}^{\bar{C}_{\hat{G}}(Q)}(\bar{f}) \cong \mathrm{Ind}_{\bar{C}_{\hat{G}}(Q, f)}^{\bar{C}_{\hat{G}}(Q)}\big(k_* \bar{C}_{\hat{H}}(Q)\bar{f} \otimes_k k_* \hat{C}_H^G(Q, f)\big) \qquad 3.4.3.$$

3.5. Note that, always from [11, Theorem 3.7], if Q is a defect group of b then there is a block $\bar{h}$ of defect zero of $\hat{C}_H^G(Q, f)$ such that we have

$$\bar{\mathrm{Br}}_Q(b) = \mathrm{Tr}_{\bar{N}_{\hat{G}}(Q, \bar{f} \otimes \bar{h})}^{\bar{N}_{\hat{G}}(Q)}(\bar{f} \otimes \bar{h}) \qquad 3.5.1;$$

in this case, denoting by $S_H^G(Q, f)$ the image of $S_{\hat{G}}(Q, f)$ in $C_H^G(Q, f)$ and by $\hat{S}_H^G(Q, f)$ the corresponding k^*-subgroup of $\hat{C}_H^G(Q, f)$, it follows again from [11, Theorem 3.7] that there is a block $\bar{\ell}$ of defect zero of $\hat{S}_H^G(Q, f)$ such that

$$\bar{h}\, \mathrm{Tr}_{\hat{C}_H^G(Q, f)_{\bar{\ell}}}^{\hat{C}_H^G(Q, f)}(\bar{\ell}) = \bar{h} \quad \text{and} \quad k_* \hat{C}_H^G(Q, f)_{\bar{\ell}}\bar{\ell} = k_* \hat{S}_H^G(Q, f)\bar{\ell} \otimes_k k_* \hat{Z}_{\bar{\ell}} \qquad 3.5.2$$

where $\hat{C}_H^G(Q, f)_{\bar{\ell}}$ is the stabilizer of $\bar{\ell}$ in $\hat{C}_H^G(Q, f)$ and $\hat{Z}_{\bar{\ell}}$ a suitable k^*-group with the stabilizer $Z_{\bar{\ell}} = C_H^G(Q, f)_{\bar{\ell}}/S_H^G(Q, f)$ of $\bar{\ell}$ in Z (cf. 3.3.1) as the k^*-quotient.

3.6. Moreover, it is clear that $E_H(Q, f)$ and $C_H^G(Q, f)$ are normal subgroups of the quotient $N_G(Q, f)/Q \cdot C_H(Q)$ and therefore their converse images $\hat{E}_H(Q, f)$ and $\hat{C}_H^G(Q, f)$ in the quotient $\big(N_{\hat{G}}(Q, f) * \hat{N}_G(Q, f)^{\circ}\big)/Q \cdot C_H(Q)$ (cf. 3.2.3 and 3.4.1) still normalize each other; but, since we have

$$N_H(Q_\delta) \cap C_G(Q_\delta) = C_H(Q) \qquad 3.6.1,$$

their commutator is contained in k^*; hence, $E_H(Q,f)$ also acts on $\hat{C}_H^G(Q,f)$, acting trivially on $C_H^G(Q,f)$ and on $\hat{S}_H^G(Q,f)$. In particular, if Q is a defect group of b then $E_H(Q,f)$ fixes $\bar{\ell}$ and therefore it acts on the k^*-group $\hat{Z}_{\bar{\ell}}$, acting trivially on $Z_{\bar{\ell}}$.

3.7. But, the action of $C_G(Q,f)$ on $\hat{E}_G(Q,f)$ determines an injective group homomorphism (cf. 3.2.4)

$$Z \longrightarrow \mathrm{Hom}\big(E_H(Q,f),k^*\big) \qquad\qquad 3.7.1.$$

Hence, since Z is Abelian, if Q is a defect group of b then the action of $E_H(Q,f)$ on $\hat{Z}_{\bar{\ell}}$ induces a surjective group homomorphism

$$E_H(Q,f) \longrightarrow \mathrm{Hom}(Z_{\bar{\ell}},k^*) \qquad\qquad 3.7.2;$$

in this case, since $Z_{\bar{\ell}}$ is an Abelian p'-group and we have

$$Z(k_*\hat{Z}_{\bar{\ell}}) = k_*Z(\hat{Z}_{\bar{\ell}}) \cong k\breve{Z}_{\bar{\ell}} \qquad\qquad 3.7.3$$

where $\breve{Z}_{\bar{\ell}}$ denotes the image of $Z(\hat{Z}_{\bar{\ell}})$ in the k^*-quotient of $\hat{Z}_{\bar{\ell}}$, the group $E_H(Q,f)$ acts transitively on the set of blocks of $\hat{Z}_{\bar{\ell}}$ and, in particular, $\bar{\ell}$ is primitive in $Z\big(k_*\hat{C}_H^G(Q,f)_{\bar{\ell}}\big)^{E_H(Q,f)}$.

Proposition 3.8. *With the notation above, b belongs to $k_*\big(\hat{H}\cdot S_{\hat{G}}(Q,f)\big)$ and it is a block of $\hat{H}\cdot S_{\hat{G}}(Q,f)$.*

Proof: It follows from [10, Proposition 15.10] that b already belongs to $k_*\big(\hat{H}\cdot C_{\hat{G}}(Q,f)\big)$ and that it is a block of $\hat{H}\cdot C_{\hat{G}}(Q,f)$; thus, with the notation above, we may assume that

$$\hat{G} = \hat{H}\cdot C_{\hat{G}}(Q,f) \quad\text{and}\quad C_{\hat{H}}(Q) = S_{\hat{G}}(Q,f) \qquad\qquad 3.8.1;$$

in this case, since $\hat{G}/\hat{H}$ is a p'-group, it follows from [10, Proposition 15.9] that Q is necessarily a defect group of b.

Consequently, since we have $S_H^G(Q,f) = \{1\}$ and $C_{\hat{G}}(Q,f) = C_{\hat{G}}(Q)$ (cf. 3.8.1), it follows from 3.7 above that the unity element is primitive in the k-algebra $Z\big(k_*\hat{C}_H^G(Q)\big)^{E_H(Q,f)}$; but, we have (cf. 3.4.2)

$$Z\big(k_*\bar{C}_{\hat{G}}(Q)\bar{f}\big)^{N_{\hat{H}}(Q,f)} \cong Z\big(k_*\hat{C}_H^G(Q)\big)^{E_H(Q,f)} \qquad\qquad 3.8.2;$$

hence, the idempotent $\bar{\mathrm{Br}}_Q(c) = \mathrm{Tr}_{N_{\hat{H}}(Q,f)}^{N_{\hat{H}}(Q)}(\bar{f})$ is also primitive in the k-algebra $Z\big(k_*\bar{C}_{\hat{G}}(Q)\big)^{N_{\hat{G}}(Q)}$, which forces $\bar{\mathrm{Br}}_Q(b) = \bar{\mathrm{Br}}_Q(c)$. Since this applies to *any* block b' of $\hat{G}$ such that $b'c = b'$, we actually get $b = c$.

3.9. From now on, we assume that $\hat{G}/\hat{H}$ is a p'-group; in particular, it follows from [10, Proposition 15.9] that Q is necessarily a defect group of b; then, it follows from [10, Lemma 15.16] that the local point δ of Q on $k_*\hat{H}$ in 3.3 above splits into a set $\{(\delta,\varphi)\}_{\varphi \in \mathcal{P}(k_*\hat{C}_H^G(Q,f))}$ of local points of Q on $k_*\hat{G}$. Moreover, the blocks $\bar{h}$ of $\hat{C}_H^G(Q,f)$ and $\bar{\ell}$ of $\hat{S}_H^G(Q,f)$ respectively determine points φ of $k_*\hat{C}_H^G(Q,f)$ and ψ of $k_*\hat{S}_H^G(Q,f)$, and it is quite clear that we have (cf. 3.4.3)

$$(k_*\hat{G})(Q_{(\delta,\varphi)}) \cong \mathrm{Ind}_{\bar{C}_{\hat{G}}(Q,f)_\varphi}^{\bar{C}_{\hat{G}}(Q)} \left(k_*\bar{C}_{\hat{H}}(Q)\bar{f} \otimes_k \left(k_*\hat{C}_H^G(Q,f)\right)(\varphi) \right) \qquad 3.9.1;$$

then, setting $Z_{\bar{\ell}} = Z_\psi$, we also get a point θ of $k_*\hat{Z}_\psi$ such that (cf. 2.5.2)

$$\left(k_*\hat{C}_H^G(Q,f)\right)(\varphi) \cong \mathrm{Ind}_{C_H^G(Q,f)_\psi}^{C_H^G(Q,f)} \left(\left(k_*\hat{S}_H^G(Q,f)\right)(\psi) \otimes_k (k_*\hat{Z}_\psi)(\theta) \right) \qquad 3.9.2;$$

Denote by $\check{Z}_\psi$ the image of $Z(\hat{Z}_\psi)$ in Z_ψ and consider the action of Z_ψ on B defined in 3.3 above; choosing an idempotent i in the point (δ,φ), our next result shows how to compute the *source algebra* $A = i(k_*\hat{G})i$ of b from the Q-interior algebra $B^{\check{Z}_\psi}$.

Theorem 3.10. *With the notation above, assume that $\hat{G}/\hat{H}$ is a p'-group. Then the image of $\hat{E}_H(Q_\delta)^{\check{Z}_\psi}$ in $E_H(Q_\delta)$ coincides with the intersection $E_H(Q_\delta) \cap E_G(Q_{(\delta,\varphi)})$, this equality can be lifted to a k^*-group isomorphism from $\hat{E}_H(Q_\delta)^{\check{Z}_\psi}$ to the converse image of this intersection in $\hat{E}_G(Q_{(\delta,\varphi)})$, and we have a Q-interior algebra isomorphism*

$$A \cong B^{\check{Z}_\psi} \otimes_{\hat{E}_H(Q_\delta)^{\check{z}_\psi}} \hat{E}_G(Q_{(\delta,\varphi)}) \qquad 3.10.1.$$

.

Proof: For any $\hat{x},\hat{y} \in N_{\hat{G}}(Q,f)$, it is clear that the "difference" between $\hat{x}(a_{\hat{x}})^{-1}\hat{y}(a_{\hat{y}})^{-1}$ and $\hat{x}\hat{y}(a_{\hat{x}\hat{y}})^{-1}$ belongs to $\left((k_*\hat{H})^Q\right)^*$, and therefore the union

$$\hat{X} = \bigcup_{\hat{x} \in N_{\hat{G}}(Q,f)} \hat{x}(a_{\hat{x}})^{-1} \cdot (B^Q)^* \qquad 3.10.2$$

is a k^*-subgroup of $N_{D^*}(Q \cdot j)$; thus, since we have

$$(k_*\hat{H})^Q \cap N_{\hat{G}}(Q,f) = C_{\hat{H}}(Q) \qquad 3.10.3,$$

we obtain the exact sequence

$$1 \longrightarrow (B^Q)^* \longrightarrow \hat{X} \longrightarrow N_{\hat{G}}(Q,f)/C_{\hat{H}}(Q) \longrightarrow 1 \qquad 3.10.4;$$

moreover, since j is primitive in B^Q, we still have the exact sequence

$$1 \longrightarrow Q \cdot \left(j + J(B^Q)\right) \longrightarrow X \longrightarrow N_{\hat{G}}(Q,f)/Q \cdot C_{\hat{H}}(Q) \longrightarrow 1 \qquad 3.10.5$$

where, as usual, X denotes the k^*-quotient of $\hat{X}$.

Since the quotient $\bar{N} = N_{\hat{G}}(Q,f)/Q{\cdot}C_{\hat{H}}(Q)$ is a p'-group, this sequence is split and, actually, all the splittings are conjugate [2, Lemma 3.3 and Proposition 3.5]; thus, denoting by $\hat{\bar{N}}$ the converse image in $\hat{X}$ of a lifting of $\bar{N}$ to X, it is easily checked that the k^*-quotient of $B \cap \hat{\bar{N}}$ is isomorphic to $E_H(Q,f)$ and therefore we may assume that (cf. 2.14.3)

$$B \cap \hat{\bar{N}} = l_\delta\big(\hat{E}_H(Q,f)\big) \qquad\qquad 3.10.6;$$

then, up to suitable identifications, isomorphism 3.3.1 determines a Q-interior algebra isomorphism

$$D \cong B \otimes_{\hat{E}_H(Q,f)} \hat{\bar{N}} \qquad\qquad 3.10.7.$$

Moreover, identifying $C_H^G(Q,f)$ with its image in $\bar{N}$, it is clear that $C_H^G(Q,f)$ centralizes $Q{\cdot}j$; further, the action on B of an element of $K_H^G(Q,f)$ coincides with the action of some element in $j + J(B^Q)$ and thus $K_H^G(Q,f)$ acts trivially on B. On the other hand, since the group of fixed points $\bar{N}^Q$ of Q on $\bar{N}$ coincides with $C_H^G(Q,f)$ and since from isomorphism 3.10.7 we clearly get

$$D(Q) \cong B(Q) \otimes_{\hat{E}_H(Q,f)} \hat{\bar{N}}^Q \qquad\qquad 3.10.8,$$

it follows from isomorphism 3.4.2 that we also get a k^*-isomorphism

$$\hat{\bar{N}}^Q \cong \hat{C}_H^G(Q,f) \qquad\qquad 3.10.9.$$

Firstly assume that $\hat{G} = \hat{H}{\cdot}S_{\hat{G}}(Q,f)$; in this case, we have $Z = \{1\}$ and, according to 3.6, isomorphism 3.10.8 above becomes

$$D \cong B \otimes_k k_* \hat{K}_H^G(Q,f) \qquad\qquad 3.10.10;$$

hence, we may assume that $i = j \otimes \ell$ for some primitive idempotent ℓ in the k-algebra $k_* \hat{K}_H^G(Q,f)$; then, i centralizes B and the multiplication by i determines a Q-interior algebra isomorphism $B \cong A$; in particular, we get (cf. 2.11.2)

$$\hat{E}_H(Q_\delta) \cong \hat{F}_B(Q_\delta)^\circ \cong \hat{F}_A(Q_{(\delta,\varphi)})^\circ \cong \hat{E}_G(Q_{(\delta,\varphi)}) \qquad\qquad 3.10.11.$$

Consequently, in order to prove the theorem we may assume that $\hat{H}$ contains $S_{\hat{G}}(Q,f)$ and, in this case, firstly assume that $\hat{G} = \hat{H}{\cdot}C_{\hat{G}}(Q,f)$; then, we have $K_H^G(Q,f) = \{1\}$, $\bar{\ell} = 1$ and $\psi = \{1\}$, and isomorphism 3.10.7 above becomes (cf. 3.6)

$$D \cong B \otimes_{k^*} \hat{Z} \qquad\qquad 3.10.12;$$

in particular, D^Q contains the k-algebra $k_* \hat{Z}$; moreover, since $\hat{E}_H(Q,f)$ and $\hat{Z}$ normalize each other (cf. 3.6), $\hat{E}_H(Q,f)$ normalizes the k-subalgebra $k_* \hat{Z}$ of D and, according to 3.7 above, it acts transitively on the set of primitive

idempotents of $Z(k_*\hat{Z})$; but, it is clear that we have $Z(k_*\hat{Z}) = k_*Z(\hat{Z})$ and that its primitive idempotents have the form

$$e_\theta = \frac{1}{|\check{Z}|} \cdot \sum_{\hat{w}} \theta(\hat{w}) \cdot \hat{w}^{-1} \qquad\qquad 3.10.13$$

where $\hat{w} \in Z(\hat{Z})$ runs over a set of representatives for $\check{Z}$ and $\theta : Z(\hat{Z}) \to k^*$ is a k^*-group homomorphism; hence, choosing such a k^*-group homomorphism θ, it follows from [11, Proposition 3.2] that we have

$$D \cong \mathrm{Ind}_{\hat{E}_H(Q,f)_\theta}^{\hat{E}_H(Q,f)} (C \otimes_k k_*\hat{Z}e_\theta) \qquad\qquad 3.10.14$$

where $\hat{E}_H(Q,f)_\theta$ denotes the stabilizer of θ in $\hat{E}_H(Q,f)$ and C the centralizer of the simple k-algebra $k_*\hat{Z}e_\theta$ in $e_\theta De_\theta$.

On the one hand, since the action of $\hat{E}_H(Q,f)$ on $\hat{Z}$ determines a homomorphism from $E_H(Q,f)$ to the group $\mathrm{Hom}(Z,k^*)$ which is Abelian, $\hat{E}_H(Q,f)_\theta$ is normal in $\hat{E}_H(Q,f)$ and therefore $\hat{E}_H(Q,f)_\theta$ coincides with $\hat{E}_H(Q,f)^{\check{Z}}$. On the other hand, since Z is a p'-group, an elementary computation shows that

$$e_\theta(B \otimes_{k^*} \hat{Z})e_\theta = B^{\check{Z}} \otimes_k k_*\hat{Z}e_\theta \qquad\qquad 3.10.15$$

and therefore we get $C = B^{\check{Z}}$. In particular, since the unity element j is primitive in $(B^{\check{Z}})^Q$, up to suitable identifications, in isomorphism 3.10.14 above we may assume that $\varphi = \theta$ and $i = 1 \otimes (j \otimes e_\theta) \otimes 1$, so that we obtain a Q-interior algebra isomorphism $A \cong B^{\check{Z}}$; moreover, once again because of Z is a p'-group, we get (cf. 2.11.2)

$$\hat{E}_H(Q_\delta)^{\check{Z}} \cong \left(\hat{F}_B(Q_\delta)^\circ\right)^{\check{Z}} \cong \hat{F}_A(Q_{(\delta,\varphi)})^\circ \cong \hat{E}_G(Q_{(\delta,\varphi)}) \qquad\qquad 3.10.16.$$

Finally, in order to prove the theorem we may assume that $\hat{H}$ contains $C_{\hat{G}}(Q,f)$; then, we have $K_H^G(Q,f) = \{1\} = C_H^G(Q,f)$, $Z = \{1\}$, $\bar{\ell} = 1 = \bar{h}$ and $\psi = \{1\} = \varphi$, and in particular we get a group isomorphism

$$\bar{N} = N_{\hat{G}}(Q,f)/Q \cdot C_{\hat{H}}(Q,f) \cong E_G(Q_{(\delta,\{1\})}) \qquad\qquad 3.10.17.$$

In this case we claim that $i = j$; indeed, it is clear that the multiplication by B on the left and the action of Q by conjugation endows D with a $B \rtimes Q$-module structure and, since the idempotent j is primitive in B^Q, equality 3.3.1 provides a direct sum decomposition of D on $B \rtimes Q$-modules. More explicitly, note that B is an indecomposable $B \rtimes Q$-module since we have $\mathrm{End}_{B \rtimes Q}(B) = B^Q$; but, for any $\hat{x} \in \mathcal{X}$, the inversible element $\hat{x}(a_{\hat{x}})^{-1}j$ of D together with the action of $\hat{x}$ on Q determine an automorphism $g_{\hat{x}}$ of $B \rtimes Q$;

thus, equality 3.3.1 provides the following direct sum decomposition on inde-composable $B \rtimes Q$-modules

$$D \cong \bigoplus_{\hat{x} \in \mathcal{X}} \mathrm{Res}_{g_{\hat{x}}}(B) \qquad\qquad 3.10.18.$$

Moreover, we claim that the $B \rtimes Q$-modules $\mathrm{Res}_{g_{\hat{x}}}(B)$ and $\mathrm{Res}_{g_{\hat{x}'}}(B)$ for $\hat{x}, \hat{x}' \in \mathcal{X}$ are isomorphic if and only if $\hat{x} = \hat{x}'$; indeed, a $B \rtimes Q$-module isomorphism

$$\mathrm{Res}_{g_{\hat{x}}}(B) \cong \mathrm{Res}_{g_{\hat{x}'}}(B) \qquad\qquad 3.10.19$$

is necessarily determined by the multiplication on the right by an inversible element a of B fulfilling $u^{\hat{x}} \cdot a = a \cdot u^{\hat{x}'}$ or, equivalently, $(u \cdot j)^a = u^{\hat{x}^{-1}\hat{x}'} \cdot j$ for any $u \in Q$, which amounts to saying that the automorphism of Q deter-mined by $\hat{x}^{-1}\hat{x}' \in N_{\hat{G}}(Q, f)$ is a B-*fusion* (cf. 2.7) from Q_δ to Q_δ [6, Propo-sition 2.12]; but, it follows from [6, Proposition 2.14 and Theorem 3.1] that we have

$$F_B(Q_\delta) = E_H(Q, f) \qquad\qquad 3.10.20\,;$$

then, isomorphism 3.10.17 implies that $\hat{x}^{-1}\hat{x}'$ belongs to $N_{\hat{H}}(Q, f)$, so that we still have $\hat{x} = \hat{x}'$.

On the other hand, it is clear that Di is a direct summand of D as $B \rtimes Q$-modules and therefore there is $\hat{x} \in \mathcal{X}$ such that $\mathrm{Res}_{g_{\hat{x}}}(B)$ is a direct summand of the $B \rtimes Q$-module Di; but, it follows from [6, Proposition 2.14] that we have

$$F_D(Q_{(\delta,\{1\})}) = F_A(Q_{(\delta,\{1\})}) = E_G(Q_{(\delta,\{1\})}) \qquad\qquad 3.10.21$$

and therefore, once again applying [6, Proposition 2.12], for any element $\hat{y}$ in $N_{\hat{G}}(Q_{(\delta,\{1\})})$ there is an inversible element $d_{\hat{y}}$ in D fulfilling

$$(u \cdot i)^{d_{\hat{y}}} = u^y \cdot i \qquad\qquad 3.10.22$$

for any $u \in Q$; then, for any $\hat{x}' \in \mathcal{X}$, it is clear that $Di = Did_{\hat{x}^{-1}\hat{x}'}$ has a direct summand isomorphic to $\mathrm{Res}_{g_{\hat{x}'}}(B)$, which forces the equality of the dimensions of Di and D, proving our claim.

Consequently, from isomorphism 3.10.17 the Q-interior algebra isomor-phism 3.10.7 becomes

$$A \cong B \otimes_{\hat{E}_H(Q_\delta)} E_G(\widehat{Q_{(\delta,\{1\})}}) \qquad\qquad 3.10.23,$$

for a suitable k^*-group $E_G(\widehat{Q_{(\delta,\{1\})}})$ with k^*-quotient $E_G(Q_{(\delta,\{1\})})$, and then it easily follows from 2.14.1 that we have

$$\hat{E}_H(Q_\delta) \subset E_G(\widehat{Q_{(\delta,\{1\})}}) \cong \hat{E}_G(Q_{(\delta,\{1\})}) \qquad\qquad 3.10.24.$$

We are done.

3.11. As a matter of fact, this theorem implies [10, Corollary 15.20] *without assuming condition* [10, 15.17.1], as we show in the next result.

Corollary 3.12. *With the notation above, assume that $\hat{G}/\hat{H}$ is a p'-group. Let $\hat{G}'$ be a k^*-subgroup of $\hat{G}$ containing $\hat{H}$, b' a block of $\hat{G}'$ such that $\mathrm{Br}_Q(b') \neq 0$, and φ' a point of the k-algebra $k_* C_{\hat{H}}^{\hat{G}'}(Q, f)$ such that $Q_{(\delta, \varphi')}$ is a defect pointed group of b'. If we have $E_{G'}(Q_{(\delta, \varphi')}) = E_G(Q_{(\delta, \varphi)})$ and this equality can be lifted to a k^*-group isomorphism $\hat{E}_{G'}(Q_{(\delta, \varphi')}) \cong \hat{E}_G(Q_{(\delta, \varphi)})$ then the Frobenius Q-category $\mathcal{F}' = \mathcal{F}_{(b', \hat{G}')}$ coincides with $\mathcal{F}$, we have a natural isomorphism $\widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}} \cong \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}$ inducing an $\mathcal{O}$-module isomorphism*

$$\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F}', \widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}) \qquad 3.12.1,$$

and the restrictions to the respective source algebras induce an $\mathcal{O}$-module isomorphism

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\hat{G}', b') \qquad 3.12.2.$$

Proof: Since we assume that $E_{G'}(Q_{(\delta, \varphi')}) = E_G(Q_{(\delta, \varphi)})$, we have

$$E_H(Q_\delta) \cap E_{G'}(Q_{(\delta, \varphi')}) = E_H(Q_\delta) \cap E_G(Q_{(\delta, \varphi)}) \qquad 3.12.3$$

and therefore it follows from Theorem 3.10 that we still have a k^*-group isomorphism

$$\hat{E}_H(Q_\delta)^{\check{Z}'_{\psi'}} \cong \hat{E}_H(Q_\delta)^{\check{Z}_\psi} \qquad 3.12.4$$

which actually forces $\check{Z}'_{\psi'} \cong \check{Z}_\psi$ (cf. 3.6 and 3.7) and $B^{\check{Z}'_{\psi'}} \cong B^{\check{Z}_\psi}$. Thus, denoting by A' a *source algebra* of the block b', always from Theorem 3.10 we obtain Q-interior algebra isomorphisms

$$
\begin{aligned}
A' &\cong B^{\check{Z}'_{\psi'}} \otimes_{\hat{E}_H(Q_\delta)^{\check{z}'_{\psi'}}} \hat{E}_{G'}(Q_{(\delta, \varphi')}) \\
&\qquad \qquad \wr\| \\
B^{\check{Z}'_{\psi'}} &\otimes_{\hat{E}_H(Q_\delta)^{\check{z}_\psi}} \hat{E}_G(Q_{(\delta, \varphi)}) \quad \cong \quad A
\end{aligned}
\qquad 3.12.5.
$$

But, it follows from [6, Theorem 3.1] and from [8, Proposition 6.21] that $\mathcal{F}$ and $\mathcal{F}'$, $\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}$ and $\widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}$, $\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$ and $\mathcal{G}_k(\mathcal{F}', \widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}})$, and $\mathcal{G}_k(\hat{G}, b)$ and $\mathcal{G}_k(\hat{G}', b')$ are completely determined from the respective source algebras A of b and A' of b'. Thus, the isomorphism $A \cong A'$ forces the equality $\mathcal{F} = \mathcal{F}'$ and all the isomorphisms. We are done.

4. Reduction of the question (Q)

4.1. From now on, we prove Theorem 1.6 by revising all the contents of [10, Chap. 16]. The point is that there all the reduction arguments depend on condition [10, 16.22.1] only thoughout condition [10, 15.17.1] in [10, Corollary 15.20]; since this condition has been removed in Corollary 3.12 above, obtaining the same conclusion, it is possible to remove condition [10, 16.22.1]

in all the statements of [10, Chap. 16], proving Theorem 1.6. We revise step by step, avoiding as far as possible to repeat proofs in the first part of the proof; from 4.11 on, we have to replace the corresponding part in [10, Chap. 16] by new arguments.

4.2. Let $\hat{G}$ be a k^*-group with finite k^*-quotient G and b a block of $\hat{G}$; from [10, Proposition 16.6] we may assume that, for any nontrivial characteristic k^*-subgroup $\hat{N}$ of $\hat{G}$, any block c of $\hat{N}$ such that $bc \neq 0$ is $\hat{G}$-stable; then, from [10, Proposition 16.7] we may assume that, for any nontrivial characteristic k^*-subgroup $\hat{N}$ of $\hat{G}$, any block c such that $bc = b$ has a nontrivial defect group.

4.3. From now on, we assume that, for any nontrivial characteristic k^*-subgroup $\hat{N}$ of $\hat{G}$, any block c of $\hat{N}$ such that $bc \neq 0$ is $\hat{G}$-stable and has a nontrivial defect group, which forces $\mathbb{O}_{p'}(\hat{G}) = k^*$. Then, from [10, Proposition 16.8] we may assume that the quotient $\hat{G}/C_{\hat{G}}\big(Z\big(\mathbb{O}_p(\hat{G})\big)\big)$ is a cyclic p'-group and, moreover, from [10, Proposition 16.9] we may assume that we actually have $\mathbb{O}_p(\hat{G}) = \{1\}$. Consequently, from now on we also assume that

$$\mathbb{O}_{p'}(\hat{G}) = k^* \quad \text{and} \quad \mathbb{O}_p(\hat{G}) = \{1\} \qquad\qquad 4.3.1.$$

4.4. Then, it is well-known that the product H of all the minimal nontrivial normal subgroups of G is a *characteristic* subgroup of G isomorphic to a direct product

$$H \cong \prod_{i \in I} H_i \qquad\qquad 4.4.1$$

of a finite family of noncommutative simple groups H_i of order divisible by p [3, Theorem 1.5]. Denoting by $\hat{H}$ and by $\hat{H}_i$ the respective converse images of H and H_i in $\hat{G}$, it is quite clear that

$$\hat{H} = \widehat{\prod}_{i \in I} \hat{H}_i \quad \text{and} \quad C_{\hat{G}}(\hat{H}) = k^* \qquad\qquad 4.4.2$$

where $\widehat{\prod}_{i \in I} \hat{H}_i$ denotes the obvious *central product* of the family of k^*-groups $\hat{H}_i$ over k^*; moreover, since this decomposition is unique, the action of $\mathrm{Aut}_{k^*}(\hat{G})$ on $\hat{H}$ induces an $\mathrm{Aut}_{k^*}(\hat{G})$-action on I and, denoting by $\hat{W}$ the kernel of the action of $\hat{G}$ on I, we have $\hat{H} \subset \hat{W}$ and get an injective group homomorphism

$$\hat{W}/\hat{H} \longrightarrow \prod_{i \in I} \mathrm{Out}_{k^*}(\hat{H}_i) \qquad\qquad 4.4.3;$$

thus, admitting the announced *Classification of the Finite Simple Groups*, the quotient $\hat{W}/\hat{H}$ is *solvable*.

4.5. Let c be the block of $\hat{H}$ such that $cb = b$ and (P, e) a maximal Brauer $(b, \hat{G})$-pair; setting $Q = P \cap \hat{H}$, it follows from [10, Proposition 15.9] that Q is a defect group of c and that there is a block f of $C_{\hat{H}}(Q)$ such that we have $e\mathrm{Br}_P(f) \neq 0$ and that (Q, f) is a maximal Brauer $(c, \hat{H})$-pair; then, we consider the Frobenius P- and Q-categories [10, 3.2]

$$\mathcal{F} = \mathcal{F}_{(b, \hat{G})} \quad \text{and} \quad \mathcal{H} = \mathcal{F}_{(c, \hat{H})} \qquad \qquad 4.5.1.$$

Since clearly $c = \otimes_{i \in I} c_i$ where c_i is a block of $\hat{H}_i$, we have $Q = \prod_{i \in I} Q_i$ where Q_i is a defect group of c_i, and $f = \otimes_{i \in I} f_i$ where f_i is a block of $C_{\hat{H}_i}(Q_i)$ and (Q_i, f_i) is a maximal Brauer $(c_i, \hat{H}_i)$-pair.

4.6. Moreover, since we are assuming that any block involved in b of any nontrivial characteristic k^*-subgroup of $\hat{G}$ has positive defect, for any $i \in I$ the defect group Q_i is *nontrivial*; thus, since any $\mathcal{H}$-*selfcentralizing* subgroup T of Q [10, 4.8] contains $Z(Q) = \prod_{i \in I} Z(Q_i)$, $C_{\hat{G}}(T)$ centralizes $Z(Q_i) \neq \{1\}$ for any $i \in I$ and therefore we get

$$C_{\hat{G}}(T) \subset \hat{W} \qquad \qquad 4.6.1.$$

In particular, W contains $\hat{K} = \hat{H} \cdot C_{\hat{G}}(Q, f)$, which is actually a normal subgroup of $\hat{G}$ by the *Frattini argument*, and therefore the quotient $\hat{K}/\hat{H}$ is solvable (cf. 16.11). Then, from [10, Proposition 16.15] we may assume that this quotient has *p-solvable length* 1 and that $\hat{G}/\hat{K}$ is a cyclic p'-group; going further, from [10, Proposition 16.19] we actually may assume that $\hat{G}/\hat{H}$ is a p'-group and that $\hat{G}/\hat{K}$ is cyclic.

4.7. Consequently, from now on we assume that $\hat{G}/\hat{H}$ is a p'-group and that $\hat{G}/\hat{K}$ is cyclic. At this point, since Corollary 3.12 holds, we can remove condition [10, 16.22], and then Proposition 16.23 in [10] becomes.

Proposition 4.8 *With the notation above, assume that $\hat{G}/\hat{H}$ is a p'-group and that $C = \hat{G}/\hat{K}$ is cyclic. Denote by δ the local point of Q on $k_* \hat{H}$ determined by f, by φ the point of $k_* \hat{C}_H^G(Q, f)$ such that (δ, φ) is the local point of $Q = P$ on $k_* \hat{G} b$ determined by e, and by $\hat{G}^\varphi$ the converse image in $\hat{G}$ of the stabilizer $C_{(\delta, \varphi)}$ of (δ, φ) in C. Then b is a block of $\hat{G}^\varphi$ and, if* (Q) *holds for $(b, \hat{G}^\varphi)$, it holds for $(b, \hat{G})$.*

Proof: According to 3.9, the pair (δ, φ) has been identified indeed with a local point of $Q = P$ on $k_* \hat{G} b$, so that e determines φ; moreover, it follows from equality 3.2.2 that C acts on the set of points of $k_* \hat{C}_H^G(Q, f)$ and therefore it makes sense to consider the converse image $\hat{G}^\varphi$ of $C_{(\delta, \varphi)}$ in $\hat{G}$.

Since $C_{\hat{G}}(Q, f) \subset \hat{K} \subset \hat{G}^\varphi$, b is also a block of $\hat{G}^\varphi$ [10, Proposition 15.10] and we have $\hat{C}_H^{G^\varphi}(Q, f) = \hat{C}_H^G(Q, f)$, so that φ is also a point of $k_* \hat{C}_H^{G^\varphi}(Q, f)$ and we have $\hat{E}_{G^\varphi}(Q_{(\delta,\varphi)}) \cong \hat{E}_G(Q_{(\delta,\varphi)})$; moreover, by the *Frattini argument*, the stabilizer $\mathrm{Aut}(\hat{G})_{(P,e)}$ of (P, e) in $\mathrm{Aut}(\hat{G})_b$ *covers* $\mathrm{Out}_{k^*}(\hat{G})_b$ and therefore we get a canonical group homomorphism

$$\mathrm{Out}_{k^*}(\hat{G})_b \longrightarrow \mathrm{Out}_{k^*}(\hat{G}^\varphi)_b \qquad 4.8.1.$$

Now, setting $\mathcal{F}^\varphi = \mathcal{F}_{(b,\hat{G}^\varphi)}$, it follows from Corollary 3.12 that we have $\mathcal{O}\mathrm{Out}(\hat{G})_b$-module isomorphisms

$$\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F}^\varphi, \widehat{\mathfrak{aut}}_{(\mathcal{F}^\varphi)^{\mathrm{nc}}}) \quad \text{and} \quad \mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\hat{G}^\varphi, b) \qquad 4.8.2.$$

Thus, if there is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}^\varphi)_b$-module isomorphism

$$\mathcal{G}_k(\mathcal{F}^\varphi, \widehat{\mathfrak{aut}}_{(\mathcal{F}^\varphi)^{\mathrm{nc}}}) \cong \mathcal{G}_k(\hat{G}^\varphi, b) \qquad 4.8.3$$

then we get an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism $\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \mathcal{G}_k(\hat{G}, b)$. We are done.

Proposition 4.9. *With the notation above, assume that $\hat{G}/\hat{H}$ is a p'-group, that $\hat{G}/\hat{K}$ is cyclic and that $\hat{G} = \hat{K} \cdot N_{\hat{G}}(Q_{(\delta,\varphi)})$. Let $\hat{x}$ be an element of $N_{\hat{G}}(Q_{(\delta,\varphi)})$ such that the image of $\hat{x}$ in $\hat{G}/\hat{K}$ is a generator of this quotient, set $\hat{G}' = \hat{H} \cdot \langle \hat{x} \rangle$ and choose a block b' of $\hat{G}'$ such that $\mathrm{Br}_Q(b') \neq 0$. If (Q) holds for $(b', \hat{G}')$ then it holds for $(b, \hat{G})$.*

Proof: Let φ' be a point of the k-algebra $k_* \hat{C}_H^{G'}(Q, f)$ such that $Q_{(\delta,\varphi')}$ is a defect pointed group of b'; since the quotient $\hat{G}'/\hat{H}$ is cyclic, it is clear that $\hat{x}$ normalizes $Q_{(\delta,\varphi')}$ and therefore we have

$$E_{G'}(Q_{(\delta,\varphi')}) = E_G(Q_{(\delta,\varphi)}) \qquad 4.9.1;$$

moreover, since the quotient $E_G(Q_{(\delta,\varphi)}) / \big(E_H(Q_\delta) \cap E_G(Q_{(\delta,\varphi)}) \big)$ is cyclic and, according to Theorem 3.10 above, the converse images of the intersection $E_H(Q_\delta) \cap E_G(Q_{(\delta,\varphi)})$ in $\hat{E}_{G'}(Q_{(\delta,\varphi')})$ and $\hat{E}_G(Q_{(\delta,\varphi)})$ admit a k^*-group isomorphism lifting the identity, it follows from Lemma 4.10 below applied to the k^*-extensions that equality 4.9.1 also can be lifted to a k^*-group isomorphism $\hat{E}_{G'}(Q_{(\delta,\varphi')}) \cong \hat{E}_G(Q_{(\delta,\varphi)})$. Consequently, it follows from Corollary 3.12 that we have canonical $\mathcal{O}$-module isomorphisms

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\hat{G}', b') \quad \text{and} \quad \mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F}', \widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}) \qquad 4.9.2.$$

Then, if (Q) holds for $(b', \hat{G}')$, it is clear that, for any block $(b'', \hat{G}'')$ isomorphic to $(b', \hat{G}')$, we can choose an $\mathcal{O}$-module isomorphism

$$\gamma_{(b'',\hat{G}'')} : \mathcal{G}_k(\hat{G}'', b'') \cong \mathcal{G}_k(\mathcal{F}'', \widehat{\mathfrak{aut}}_{\mathcal{F}''^{\mathrm{nc}}}) \qquad 4.9.3,$$

where $\mathcal{F}'' = \mathcal{F}_{(b'',\hat{G}'')}$, in such a way that these isomorphisms are compatible with the isomorphisms between these blocks. At this point, it is easily checked that the obvious composition

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\hat{G}', b') \overset{\gamma_{(b',\hat{G}')}}{\cong} \mathcal{G}_k(\mathcal{F}', \widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.9.4$$

is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism. We are done.

Lemma 4.10. *Let K be a finite group, H a normal subgroup of K such that the quotient K/H is cyclic, A a divisible Abelian group and $\hat{H}$ a central A-extension of H. Assume that the action of K on H can be lifted to an action of K on $\hat{H}$ such that we have $\mathbb{H}^2(K/H, A) = \{0\}$. Then, there exists an essentially unique A-extension $\hat{K}$ of K containing $\hat{H}$ and lifting the inclusion map $H \to K$. In particular, any automorphism τ of K stabilizing H which can be lifted to an automorphism $\hat{\sigma}$ of $\hat{H}$, can be lifted to an automorphism of $\hat{K}$ extending $\hat{\sigma}$.*

Proof: Choose a cyclic subgroup C of K such that $K = H \cdot C$ and set $D = C \cap H$; since the converse image $\hat{D}$ of D in $\hat{H}$ is split, we can choose a splitting $\theta : D \to \hat{D} \subset \hat{H}$ and, since $C \subset K$ acts on $\hat{H}$, we can consider the semidirect product $\hat{H} \rtimes C$; inside, we define the "inverse diagonal"

$$\Delta^*(D) = \{(\theta(y), y^{-1})\}_{y \in D} \qquad 4.10.1$$

and it is easily checked that $\Delta^*(D)$ is a subgroup contained in the center of $\hat{H} \rtimes C$; then, it suffices to set

$$\hat{K} = (\hat{H} \rtimes C)/\Delta^*(D) \qquad 4.10.2;$$

indeed, the structural homomorphism $\hat{H} \to \hat{H} \rtimes C$ determines an injection $\hat{H} \to \hat{K}$ lifting the inclusion $H \subset K$.

Moreover, if $\check{K}$ is an A-extension of K containing $\hat{H}$ and lifting the inclusion map $H \to K$, then $\check{K} * \hat{K}^\circ$ contains $\hat{H} * \hat{H}^\circ$ which is *canonically* isomorphic to $A \times H$ and therefore, up to suitable identifications, the quotient $(\check{K} * \hat{K}^\circ)/H$ is an A-extension of the cyclic group K/H *via* the action which is induced by the action of K on $\hat{H}$; but, we assume that $\mathbb{H}^2(K/H, A) = \{0\}$; hence, this extension is split and therefore $\check{K} * \hat{K}^\circ$ is also split or, more precisely, there is an isomorphism $\check{K} \cong \hat{K}$ inducing the identity on $\hat{H}$.

In particular, if τ is an automorphism of K stabilizing H which can be lifted to an automorphism $\hat{\sigma}$ of $\hat{H}$, it induces a group isomorphism

$$\hat{H} \rtimes C \cong \hat{H} \rtimes \tau(C) \qquad\qquad 4.10.3$$

mapping $\Delta^*(D)$ onto $\Delta^*\big(\tau(D)\big)$ and therefore it determines an isomorphism

$$\hat{K} \cong \big(\hat{H} \rtimes \tau(C)\big)/\Delta^*\big(\tau(D)\big) \qquad\qquad 4.10.4;$$

but, the right member of this isomorphism is also an A-extension of $\hat{H}$ lifting the inclusion map $H \to K$ and therefore it admits an isomorphism to $\hat{K}$ inducing the identity on $\hat{H}$. We are done.

4.11. Thus, we may assume that $\hat{G}/\hat{H}$ is a cyclic p'-group. But, we have $\hat{H} \cong \hat{\prod}_{i \in I} \hat{H}_i$ and we want to reduce our situation to the case where I has a unique element. In order to do this reduction, we will apply [10, Corollary 15.47] which forces us to move to a "bigger" situation; namely, for any $i \in I$, let us denote by K_i the image of $\hat{K}$ in $\mathrm{Aut}(\hat{H}_i)$; note that, by the very definition of $\hat{K}$ (cf. 4.6), we have

$$K_i = H_i \cdot C_{K_i}(Q_i, f_i) \qquad\qquad 4.11.1.$$

Since K_i/H_i is cyclic, it follows from Lemma 4.10 that there exists an essentially unique k^*-group $\hat{K}_i$ containing $\hat{H}_i$; set $\hat{K}^* = \hat{\prod}_{i \in I} \hat{K}_i$. Then, since K/H is cyclic, identifying K to its canonical image in $K^* = \prod_{i \in I} K_i$, it follows again from Lemma 4.10 that we can identify $\hat{K}$ with the converse image of K in $\hat{K}^*$.

4.12. Similarly, we can identify G and K^* with their image in $\mathrm{Aut}(\hat{H})$ and, in this group, we set $G^* = K^* \cdot G$; once again, since G^*/K^* is cyclic, there exists an essentially unique k^*-group $\hat{G}^*$ containing $\hat{K}^*$ and, since $\hat{G}/\hat{H}$ is cyclic, we can identify $\hat{G}$ with the converse image of G in $\hat{G}^*$; then, it is clear that

$$\hat{K}^* \cap \hat{G} = \hat{K} \quad \text{and} \quad \hat{K}^* = \hat{H} \cdot C_{\hat{G}^*}(Q, f) \qquad\qquad 4.12.1.$$

Moreover, for any $i \in I$, since the quotient

$$C_{H_i}^{K_i}(Q_i, f_i) = C_{K_i}(Q_i, f_i)/C_{H_i}(Q_i, f_i) \qquad\qquad 4.12.2$$

is cyclic, the k^*-group $\hat{C}_{H_i}^{K_i}(Q_i, f_i)$ is split and it is quite clear that we can choose a $N_G(Q, f)$-stable family of k^*-group homomorphisms

$$\hat{\varphi}_i : \hat{C}_{H_i}^{K_i}(Q_i, f_i) \longrightarrow k^* \qquad\qquad 4.12.3;$$

now, since $\hat{C}_H^{K^*}(Q, f) = \hat{C}_H^{G^*}(Q, f)$, this family determines a $N_G(Q, f)$-stable point φ^* of $k_* \hat{C}_H^{G^*}(Q, f)$ and then the pair (δ, φ^*) determines a local point

of Q on $k_*\hat{G}^*$ (cf. 3.9); it is quite clear that $Q_{(\delta,\varphi^*)}$ is a defect pointed group of a block b^* of $\hat{G}^*$ (cf. 2.9) and we set $\mathcal{F}^* = \mathcal{F}_{(b^*,G^*)}$. Now, we replace Proposition 16.25 in [10] by the following result.

Proposition 4.13. *With the notation above, assume that $\hat{G}/\hat{H}$ is a cyclic p'-group and that $\hat{G} = \hat{K}{\cdot}N_{\hat{G}}(Q_{(\delta,\varphi)})$. If* (Q) *holds for $(b^\star,\hat{G}^\star)$ then it holds for $(b,\hat{G})$.*

Proof: Since $N_G(Q,f)$ normalizes $Q_{(\delta,\varphi^*)}$, we clearly have

$$E_{G^*}(Q_{(\delta,\varphi^*)}) = E_G(Q_{(\delta,\varphi)}) \qquad\qquad 4.13.1;$$

moreover, since the quotient $E_G(Q_{(\delta,\varphi)})/\big(E_H(Q_\delta)\cap E_G(Q_{(\delta,\varphi)})\big)$ is cyclic and, according to Theorem 3.10 above, the converse images of the intersection $E_H(Q_\delta) \cap E_G(Q_{(\delta,\varphi)})$ in $\hat{E}_{G^*}(Q_{(\delta,\varphi^*)})$ and $\hat{E}_G(Q_{(\delta,\varphi)})$ admit a k^*-group isomorphism lifting the identity, it follows from Lemma 4.10 that equality 4.13.1 also can be lifted to a k^*-group isomorphism $\hat{E}_{G^*}(Q_{(\delta,\varphi^*)}) \cong \hat{E}_G(Q_{(\delta,\varphi)})$. Consequently, it follows from Corollary 3.12 that we have canonical $\mathcal{O}$-module isomorphisms

$$\mathcal{G}_k(\hat{G},b) \cong \mathcal{G}_k(\hat{G}^*,b^*) \quad \text{and} \quad \mathcal{G}_k(\mathcal{F},\widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F}^*,\widehat{\mathrm{aut}}_{(\mathcal{F}^*)^{\mathrm{nc}}}) \qquad 4.13.2.$$

Then, if (Q) holds for $(b^*,\hat{G}^*)$, it is clear that, for any block $(\bar{b}^*,\hat{G}^*)$ isomorphic to $(b^*,\hat{G}^*)$, we can choose an $\mathcal{O}$-module isomorphism

$$\gamma_{(\bar{b}^*,\hat{G}^*)} : \mathcal{G}_k(\hat{G}^*,\bar{b}^*) \cong \mathcal{G}_k(\bar{\mathcal{F}}^*,\widehat{\mathrm{aut}}_{(\bar{\mathcal{F}}^*)^{\mathrm{nc}}}) \qquad\qquad 4.13.3,$$

where $\bar{\mathcal{F}}^* = \mathcal{F}_{(\bar{b}^*,\hat{G}^*)}$, in such a way that these isomorphisms are compatible with the isomorphisms between these blocks. At this point, it is easily checked that the obvious composition

$$\mathcal{G}_k(\hat{G},b) \cong \mathcal{G}_k(\hat{G}^*,b^*) \overset{\gamma_{(b^*,\hat{G}^*)}}{\cong} \mathcal{G}_k(\mathcal{F}^*,\widehat{\mathrm{aut}}_{(\mathcal{F}^*)^{\mathrm{nc}}}) \cong \mathcal{G}_k(\mathcal{F},\widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.13.4$$

is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism. We are done.

4.14. Consequently, from now on we assume that $K = \prod_{i\in I} K_i$ and that the quotients $C = G/K$ and K_i/H_i for any $i \in I$ are cyclic p'-groups; then, we have

$$\hat{K} = \widehat{\prod}_{i\in I} \hat{K}_i \qquad\qquad 4.14.1$$

and, since b is also a block of $\hat{K}$ [10, Proposition 15.10], we have $b = \otimes_{i\in I} b_i$ for a suitable block b_i of $\hat{K}_i$ for any $i \in I$; moreover, we set $\mathcal{K} = \mathcal{F}_{(b,\hat{K})}$ and $\mathcal{K}^i = \mathcal{F}_{(b_i,\hat{K}_i)}$ for any $i \in I$. Since $\hat{G} = \hat{K}{\cdot}N_{\hat{G}}(Q,f)$, it is clear that

$$C \cong \hat{G}/\hat{K} \cong N_{\hat{G}}(Q,f)/N_{\hat{K}}(Q,f) \cong \mathcal{F}(Q)/\mathcal{K}(Q) \qquad 4.14.2;$$

then, for any subgroup D of C, we denote by ${}^D\hat{K}$ the converse image of D in $\hat{G}$ and set ${}^D\mathcal{K} = \mathcal{F}_{(b,\,{}^D\hat{K})}$. Recall that we respectively denote by $\mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G},b)$ and $\mathcal{R}_\mathcal{K}\mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$ the intersection of the kernels of all the respective $\mathcal{O}$-module homomorphisms determined by the restriction

$$\mathcal{G}_k(\hat{G},b) \to \mathcal{G}_k({}^D\hat{K},b) \quad\text{and}\quad \mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \to \mathcal{G}_k({}^D\mathcal{K},\widehat{\mathfrak{aut}}_{({}^D\mathcal{K})^{\mathrm{nc}}}) \quad 4.14.3$$

where D runs over the set of proper subgroups of C.

4.15. It is clear that the quotient $C = G/K$ acts on I; if I decomposes on a disjoint union of two nonempty C-stable subsets I' and I'' then, setting

$$\hat{K}' = \prod_{i'\in I'}^{\hat{}} \hat{K}_{i'} \quad\text{and}\quad \hat{K}'' = \prod_{i''\in I''}^{\hat{}} \hat{K}_{i''} \qquad 4.15.1,$$

it follows again from Lemma 4.10 that there exist essentially unique k^*-groups $\hat{G}'$ and $\hat{G}''$, respectively containing and normalizing $\hat{K}'$ and $\hat{K}''$, such that

$$\hat{G}'/\hat{K}' \cong C \cong \hat{G}''/\hat{K}'' \quad\text{and}\quad \hat{G}' \hat{\times}_C \hat{G}'' \cong \hat{G} \qquad 4.15.2.$$

Moreover, setting $b' = \otimes_{i'\in I'} b_{i'}$ and $b'' = \otimes_{i''\in I''} b_{i''}$, it follows from [10 Proposition 15.10] that b' and b'' are respective blocks of $\hat{G}'$ and $\hat{G}''$; we set

$$\begin{aligned} \mathcal{F}' &= \mathcal{F}_{(b',\hat{G}')} \quad\text{and}\quad \mathcal{F}'' = \mathcal{F}_{(b'',\hat{G}'')} \\ \mathcal{K}' &= \mathcal{F}_{(b',\hat{K}')} \quad\text{and}\quad \mathcal{K}'' = \mathcal{F}_{(b'',\hat{K}'')} \end{aligned} \qquad 4.15.3.$$

Note that, for any subgroup D of C, we have an analogous situation with respect to the converse images ${}^D\hat{K}$, ${}^D\hat{K}'$ and ${}^D\hat{K}''$ of D in $\hat{G}$, $\hat{G}'$ and $\hat{G}''$.

Proposition 4.16 *With the notation above, assume that* $\mathrm{Aut}_{k^*}(\hat{G})_b$ *stabilizes* I' *and* I''. *If* (Q) *holds for* $(b',{}^D\hat{K}')$ *and* $(b'',{}^D\hat{K}'')$ *for any subgroup* D *of* C, *then it holds for* $(b,\hat{G})$.

Proof: According to our hypothesis, we have canonical group homomorphisms

$$\mathrm{Out}_{k^*}(\hat{G}')_{b'} \longleftarrow \mathrm{Out}_{k^*}(\hat{G})_b \longrightarrow \mathrm{Out}_{k^*}(\hat{G}'')_{b''} \qquad 4.16.1;$$

then, since any homomorphism from C to k^* induces k^*-group automorphisms of $\hat{G}$, $\hat{G}'$ and $\hat{G}''$ which are contained in the centers of $\mathrm{Aut}_{k^*}(\hat{G})$, $\mathrm{Aut}_{k^*}(\hat{G}')$ and $\mathrm{Aut}_{k^*}(\hat{G}'')$ respectively, it follows from [10, Corollary 15.47] that, setting $\hat{\mathcal{R}} = \mathcal{R}\mathcal{G}_k(C)$, we have $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphisms

$$\begin{aligned} \mathcal{R}_{\hat{K}'}\mathcal{G}_k(\hat{G}',b') \otimes_{\hat{\mathcal{R}}} \mathcal{R}_{\hat{K}''}\mathcal{G}_k(\hat{G}'',b'') &\cong \mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G},b) \\ \mathcal{R}_{\mathcal{K}'}\mathcal{G}_k(\mathcal{F}',\widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}) \otimes_{\hat{\mathcal{R}}} \mathcal{R}_{\mathcal{K}''}\mathcal{G}_k(\mathcal{F}'',\widehat{\mathfrak{aut}}_{\mathcal{F}''^{\mathrm{nc}}}) &\cong \mathcal{R}_\mathcal{K}\mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \end{aligned} \qquad 4.16.2.$$

Moreover, assume that we have $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}')_{b'}$- and $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}'')_{b''}$-module isomorphisms

$$\mathcal{G}_k(\hat{G}',b') \cong \mathcal{G}_k(\mathcal{F}',\widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}}) \quad \text{and} \quad \mathcal{G}_k(\hat{G}'',b'') \cong \mathcal{G}_k(\mathcal{F}'',\widehat{\mathfrak{aut}}_{\mathcal{F}''^{\mathrm{nc}}}) \quad 4.16.3;$$

since the restriction induces *compatible* $\mathcal{G}_k(C)$-module structures on all the members of these isomorphisms [10, 15.21 and 15.33], it follows from [10, 15.23.2 and 15.37.1] that we still have $\hat{\mathcal{R}}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphisms

$$\mathcal{R}_{\hat{K}'}\mathcal{G}_k(\hat{G}',b') \cong \mathcal{R}_{\mathcal{K}'}\mathcal{G}_k(\mathcal{F}',\widehat{\mathfrak{aut}}_{\mathcal{F}'^{\mathrm{nc}}})$$
$$\mathcal{R}_{\hat{K}''}\mathcal{G}_k(\hat{G}'',b'') \cong \mathcal{R}_{\mathcal{K}''}\mathcal{G}_k(\mathcal{F}'',\widehat{\mathfrak{aut}}_{\mathcal{F}''^{\mathrm{nc}}})$$
$$4.16.4.$$

Then, from isomorphisms 4.16.2 we get an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G},b) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.16.5.$$

Consequently, according to our hypothesis, for any subgroup D of C we have an $\mathcal{O}\mathrm{Out}_{k^*}(^D\hat{K})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(^D\hat{K},b) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(^D\mathcal{K},\widehat{\mathfrak{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}}) \qquad 4.16.6;$$

but, since $\mathrm{Aut}_{k^*}(\hat{G})_b$ stabilizes $\hat{K}$, we have evident group homomorphisms

$$C \longrightarrow \mathrm{Out}_{k^*}(^{\bar{D}}\hat{K})_b \longleftarrow \mathrm{Aut}_{k^*}(\hat{G})_b \qquad 4.16.7$$

and it is clear that the image of $\mathrm{Aut}_{k^*}(\hat{G})_b$ contains and normalizes the image of C; hence, we still have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(^D\hat{K},b)^C \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(^D\mathcal{K},\widehat{\mathfrak{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}})^C \qquad 4.16.8.$$

Then, it follows from [10, 15.23.4 and 15.38.1] that the direct sum of isomorphisms 4.16.8 when D runs over the set of subgroups of C supplies an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism $\mathcal{G}_k(\hat{G},b) \cong \mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$. We are done.

4.17 From now on, we assume that the group $\mathrm{Aut}_{k^*}(\hat{G})_b$ acts transitively on I; in particular, it acts transitively on the set of C-orbits of I and, for any C-orbit O we consider the k^*-group and the block

$$\hat{K}^O = \hat{\prod}_{i\in O}\hat{K}_i \quad \text{and} \quad b^O = \bigotimes_{i\in O} b_i \qquad 4.17.1;$$

once again, it follows from Lemma 4.10 that there exists an essentially unique k^*-group $\hat{G}^O$ containing $\hat{K}^O$ and fulfilling $C \cong \hat{G}^O/\hat{K}^O$; then, it follows from [10, Proposition 15.10] that b^O is also a block of $\hat{G}^O$ and we set

$$\mathcal{F}^O = \mathcal{F}_{(b^O,\hat{G}^O)} \quad \text{and} \quad \mathcal{K}^O = \mathcal{F}_{(b^O,\hat{K}^O)} \qquad 4.17.2.$$

Note that $\hat{G}$ is isomorphic to the *direct sum over C* of the family of k^*-groups $\hat{G}^O$ when O runs over the set of C-orbits of I.

Proposition 4.18 *With the notation above, assume that $\mathrm{Aut}_{k^*}(\hat{G})_b$ acts transitively on I and let O be a C-orbit of I. If (Q) holds for $\big(b^O, {}^D(\hat{K}^O)\big)$ for any subgroup D of C, then it holds for $(b, \hat{G})$.*

Proof: It is clear that the action of $\mathrm{Aut}_{k^*}(\hat{G})_b$ on I induces an action of $\mathrm{Out}_{k^*}(\hat{G})_b$ on the set $\bar{I}$ of C-orbits of I; moreover, denoting by $\mathrm{Out}_{k^*}(\hat{G})_{b,O}$ the stabilizer of O in $\mathrm{Out}_{k^*}(\hat{G})_b$, it is quite clear that the restriction induces a group homomorphism

$$\mathrm{Out}_{k^*}(\hat{G})_{b,O} \longrightarrow \mathrm{Out}_{k^*}(\hat{G}^O)_{b^O} \qquad 4.18.1\,.$$

On the other hand, it is quite clear that $\mathrm{Out}_{k^*}(\hat{G})_b$ acts transitively on the two families of $\mathcal{O}$-modules

$$\{\mathcal{G}_k(\hat{G}^{O'}, b^{O'})\}_{O' \in \bar{I}} \quad \text{and} \quad \{\mathcal{R}_{\mathcal{K}^{O'}}\mathcal{G}_k(\mathcal{F}^{O'}, \widehat{\mathrm{aut}}_{(\mathcal{F}^{O'})^{\mathrm{nc}}})\}_{O' \in \bar{I}} \qquad 4.18.2$$

and then, iterating the canonical isomorphisms in [10, Corollary 15.47] and setting $\hat{\mathcal{R}} = \mathcal{R}\mathcal{G}_k(C)$, it is not difficult to check that we have $\hat{\mathcal{R}}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphisms

$$\hat{\mathcal{R}}\mathbf{Ten}_{\mathrm{Out}_{k^*}(\hat{G})_{b,O}}^{\mathrm{Out}_{k^*}(\hat{G})_b}\big(\mathcal{R}_{\hat{K}^O}\mathcal{G}_k(\hat{G}^O, b^O)\big) \cong \mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G}, b)$$

$$\hat{\mathcal{R}}\mathbf{Ten}_{\mathrm{Out}_{k^*}(\hat{G})_{b,O}}^{\mathrm{Out}_{k^*}(\hat{G})_b}\big(\mathcal{R}_{\mathcal{K}^O}\mathcal{G}_k(\mathcal{F}^O, \widehat{\mathrm{aut}}_{(\mathcal{F}^O)^{\mathrm{nc}}})\big) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.18.3$$

where $\hat{\mathcal{R}}\mathbf{Ten}$ denotes the usual *tensor induction of $\hat{\mathcal{R}}$-modules*.

Moreover, assume that we have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}^O)_{b^O}$-module isomorphism

$$\mathcal{G}_k(\hat{G}^O, b^O) \cong \mathcal{G}_k(\mathcal{F}^O, \widehat{\mathrm{aut}}_{(\mathcal{F}^O)^{\mathrm{nc}}}) \qquad 4.18.4;$$

then, since the restriction induces compatible $\mathcal{G}_k(C)$-module structures on both members of this isomorphism [10, 15.21 and 15.33], it follows from [10, 15.23.2 and 15.37.1] that we still have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_{b,O}$-module isomorphism

$$\mathcal{R}_{\hat{K}^O}\mathcal{G}_k(\hat{G}^O, b^O) \cong \mathcal{R}_{\mathcal{K}^O}\mathcal{G}_k(\mathcal{F}^O, \widehat{\mathrm{aut}}_{(\mathcal{F}^O)^{\mathrm{nc}}}) \qquad 4.18.5.$$

Thus, from isomorphisms 4.18.3 above, we get an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G}, b) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.18.6.$$

Consequently, according to our hypothesis and possibly applying Proposition 4.16 above and [10, 15.23.2 and 15.37.1], for any subgroup D of C we have an $\mathcal{O}\mathrm{Out}_{k^*}(^D\hat{K})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k\big(^D\hat{K},b\big) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(^D\mathcal{K},\widehat{\mathfrak{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}}) \qquad 4.18.7;$$

but, since $\mathrm{Aut}_{k^*}(\hat{G})_b$ stabilizes $\hat{K}$, we have evident group homomorphisms

$$\bar{C} \longrightarrow \mathrm{Out}_{k^*}(^D\hat{K})_b \longleftarrow \mathrm{Aut}_{k^*}(\hat{G})_b \qquad 4.18.8$$

and it is clear that the image of $\mathrm{Aut}_{k^*}(\hat{G})_b$ contains and normalizes the image of C; hence, we still have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k\big(^D\hat{K},b\big)^C \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k\big(^D\mathcal{K},\widehat{\mathfrak{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}}\big)^C \qquad 4.18.9.$$

Then, it follows from [10, 15.23.4 and 15.38.1] that the direct sum of isomorphisms 4.18.9 when D runs over the set of subgroups of C supplies an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism $\mathcal{G}_k(\hat{G},b) \cong \mathcal{G}_k(\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$. We are done.

4.19. In the last step of our reduction, we assume that $C = G/K$ acts transitively on I. In this situation, we have to consider the *direct product* of groups

$$\hat{\hat{K}} = \prod_{i\in I} \hat{K}_i \qquad 4.19.1;$$

since C is cyclic and it acts on $(k^*)^I$ permuting the factors, it follows from Lemma 4.10 that there exists an essentially unique $(k^*)^I$-extension $\hat{\hat{G}}$ of G containing $\hat{\hat{K}}$. Moreover, denoting by $\nabla_{k^*}:(k^*)^I \to k^*$ the group homomorphism induced by the product in k and considering the *group algebras* of the *groups* $(k^*)^I$ and $\hat{\hat{G}}$ over k and the k-algebra homomorphism $k(k^*)^I \to k$ determined by ∇_{k^*}, it is quite clear that we have a k^*-group and a k-algebra isomorphisms

$$\hat{\hat{G}}/\mathrm{Ker}(\nabla_{k^*}) \cong \hat{G} \quad \text{and} \quad k \otimes_{k(k^*)^I} k\hat{\hat{G}} \cong k_*\hat{G} \qquad 4.19.2.$$

4.20. In particular, since $\hat{G}$ acts transitively on the family $\{\hat{K}_i\}_{i\in I}$ and, for any $i \in I$, b_i is a block of $\hat{K}_i$ (cf. 4.14), by the *Frattini argument* we get canonical group homomorphisms

$$\mathrm{Out}_{k^*}(\hat{G})_b \longrightarrow \mathrm{Out}_{k^*}(\hat{K}_i)_{b_i} \qquad 4.20.1.$$

Moreover, choose an element $i \in I$ and respectively denote by C_i, $\hat{G}_i$ and $\hat{\hat{G}}_i$ the stabilizers of i in C, $\hat{G}$ and $\hat{\hat{G}}$, which actually act trivially on I; setting

$I' = I - \{i\}$, it is clear that $\prod_{i' \in I'} \hat{K}_{i'}$ is a normal subgroup of $\hat{\hat{G}}_i$ and that the quotient

$$\hat{G}^i = \hat{\hat{G}}_i / \big(\prod_{i' \in I'} \hat{K}_{i'} \big) \qquad\qquad 4.20.2$$

is a k^*-group which contains $\hat{K}_i$ as a normal k^*-subgroup; now, any $k_* \hat{G}^i$-module M_i can be viewed as a $k\hat{\hat{G}}_i$-module, and the point is that *the tensor induction* $\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i)$ *becomes a $k_* \hat{G}$-module* Let us consider $\mathcal{RG}_k(C)$ as an $\mathcal{RG}_k(C_i)$-algebra *via* the group homomorphism mapping $c \in C$ on $c^{|I|}$.

Proposition 4.21. *With the notation above, assume that C acts transitively on I and choose an element i of I. For any $k_* \hat{G}^i$-module M_i considered as a $k\hat{\hat{G}}_i$-module, $\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i)$ becomes a $k_* \hat{G}$-module, and this correspondence induces an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism*

$$\mathcal{RG}_k(C) \otimes_{\mathcal{RG}_k(C_i)} \mathcal{R}_{\hat{K}^i} \mathcal{G}_k(\hat{G}^i, b_i) \cong \mathcal{R}_{\hat{K}} \mathcal{G}_k(\hat{G}, b) \qquad 4.21.1.$$

Proof: Recall that the *tensor induction* of M_i from $\hat{\hat{G}}_i$ to $\hat{\hat{G}}$ is the $k\hat{\hat{G}}$-module

$$\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i) = \bigotimes_{X \in \hat{\hat{G}}/\hat{\hat{G}}_i} (kX \otimes_{k\hat{\hat{G}}_i} M_i) \qquad 4.21.2,$$

where kX denotes the k-vector space over the (right-hand) $\hat{\hat{G}}_i$-class X of $\hat{\hat{G}}$, endowed with the (right-hand) $k\hat{\hat{G}}_i$-module structure determined by the multiplication on the right [10, 8.2]. It is clear that in $\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i)$ the multiplication by $(\lambda_i)_{i \in I} \in (k^*)^I$ coincides with the multiplication by $\prod_{i \in I} \lambda_i \in k^*$, so that, according to isomorphisms 4.19.2, $\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i)$ becomes a $k_* \hat{G}$-module.

Moreover, if we have $M_i = M_i' \oplus M_i''$ as $k\hat{\hat{G}}_i$-modules then we clearly get

$$\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(M_i)$$
$$= \bigoplus_{\mathcal{X}} \big(\bigotimes_{X \in \mathcal{X}} (kX \otimes_{k\hat{\hat{G}}_i} M_i') \big) \otimes_k \big(\bigotimes_{Y \in \hat{\hat{G}}/\hat{\hat{G}}_i - \mathcal{X}} (kY \otimes_{k\hat{\hat{G}}_i} M_i'') \big) \qquad 4.21.3$$

where $\mathcal{X}$ runs over the set of all the subsets of $\hat{\hat{G}}/\hat{\hat{G}}_i$; in particular, since any p'-element $\hat{\hat{x}} \in \hat{\hat{G}}$ such that $\hat{\hat{G}} = \hat{\hat{G}}_i \cdot \langle \hat{\hat{x}} \rangle$ stabilizes this direct sum but only fixes the terms labeled by $\emptyset$ and by $\hat{\hat{G}}/\hat{\hat{G}}_i$, denoting by χ_i the *modular character* of M_i, we get

$$\big(\mathrm{Ten}^{\hat{\hat{G}}}_{\hat{\hat{G}}_i}(\chi_i) \big)(\hat{\hat{x}}) = \chi_i \big(\overline{\hat{\hat{x}}^{|I|}}^i \big) \qquad 4.21.4$$

where $\overline{\hat{\hat{x}}^{|I|}}^{\,i}$ denotes the image of $\hat{\hat{x}}^{|I|} \in \hat{\hat{G}}_i$ in $\hat{G}^i$; in particular, this equality shows that the tensor induction $\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}$ induces an $\mathcal{O}$-module homomorphism from $\mathcal{G}_k(\hat{G}^i)$ to the $\mathcal{O}$-module formed by the restriction of *modular characters* of $\hat{G}$ to the set of p'-elements $\hat{x} \in \hat{G}$ such that $\hat{G} = \hat{G}_i \cdot \langle \hat{x} \rangle$.

But, by the very definition of $\mathcal{R}_{\hat{G}_i} \mathcal{G}_k(\hat{G})$ in [10, 15.22.4], $\mathcal{R}_{\hat{G}_i} \mathcal{G}_k(\hat{G})$ is isomorphic to this $\mathcal{O}$-module and this restriction is equivalent to the projection obtained from [10, 15.23.4]

$$\mathcal{G}_k(\hat{G}) \longrightarrow \mathcal{R}_{\hat{G}_i} \mathcal{G}_k(\hat{G}) \qquad\qquad 4.21.5,$$

so that, we finally get an $\mathcal{O}$-module homomorphism

$$\mathcal{G}_k(\hat{G}^i) \longrightarrow \mathcal{R}_{\hat{G}_i} \mathcal{G}_k(\hat{G}) \qquad\qquad 4.21.6;$$

more precisely, it is quite clear that this homomorphism maps $\mathcal{R}_{\hat{K}^i} \mathcal{G}_k(\hat{G}^i)$ on $\mathcal{R}_{\hat{K}} \mathcal{G}_k(\hat{G})$ and, since $\mathcal{R}_{\hat{K}^i} \mathcal{G}_k(\hat{G}^i)$ and $\mathcal{R}_{\hat{K}} \mathcal{G}_k(\hat{G})$ respectively have $\mathcal{R}\mathcal{G}_k(C_i)$- and $\mathcal{R}\mathcal{G}_k(C)$-module structures, which are compatible with homomorphism 4.21.5 above, we still get an $\mathcal{R}\mathcal{G}_k(C)$-module homomorphism

$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\hat{K}^i} \mathcal{G}_k(\hat{G}^i) \longrightarrow \mathcal{R}_{\hat{K}} \mathcal{G}_k(\hat{G}) \qquad\qquad 4.21.7$$

and we claim that it is bijective.

Indeed, if M_i is a simple $k_* \hat{G}^i$-module such that the restriction to $\hat{K}_i$ remains simple then, since $\hat{K} = \widehat{\prod}_{j \in I} \hat{K}_j$ and therefore $k_* \hat{K} \cong \otimes_{j \in I} k_* \hat{K}_j$, it is clear that the restriction of $\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}(M_i)$ to $k_* \hat{K}$ is simple too. Conversely, if M is a simple $k_* \hat{G}$-module such that the restriction to $\hat{K}$ remains simple then we necessarily have $M \cong \otimes_{j \in I} M_j$ or, more explicitly,

$$\mathrm{End}_k(M) \cong \bigotimes_{j \in I} S_j \qquad\qquad 4.21.8$$

where $S_j \cong \mathrm{End}_k(M_j)$ is generated by the image of $\hat{K}_j \subset \hat{K}$ for any $j \in I$; thus, $\hat{G}$ stabilizes the family of k-subalgebras $\{S_j\}_{j \in I}$ of $\mathrm{End}_k(M)$, and $\hat{G}_j$, which coincides with $\hat{G}_i$, stabilizes S_j for any $j \in I$; in particular, the image of any $\hat{x}_i \in \hat{G}_i$ in $\mathrm{End}_k(M)$ has the form $\otimes_{j \in I} s_j$ for suitable $s_j \in S_j$ for any $j \in I$; then, considering $\hat{G}_i$ as a quotient of $\hat{\hat{G}}_i$, it is clear that the corresponding homomorphism $\hat{\hat{G}}_i \to \mathrm{End}_k(M)$ factorizes throughout a $(k^*)^I$-extension homomorphism

$$\hat{\hat{G}}_i \longrightarrow \prod_{j \in I} S_j \qquad\qquad 4.21.9$$

and then that the corresponding homomorphism $\hat{\hat{G}}_i \to S_i$ factorizes through-out a k^*-group homomorphism $\hat{G}^i \to S_i$, so that M_i becomes a $k_*\hat{G}^i$-module which remains simple restricted to $\hat{K}_i$.

Moreover, the groups $\mathrm{Hom}(C_i, k^*)$ and $\mathrm{Hom}(C, k^*)$ respectively determine k^*-group automorphisms of $\hat{G}^i$ and $\hat{G}$, and it is elementary to check that the restriction of M *via* the k^*-group automorphism of $\hat{G}$ determined by a suitable element of $\mathrm{Hom}(C, k^*)$ coincides with $\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}(M_i)$. Now, it is not difficult to check that this correspondence induces a bijection between the set of $\mathrm{Hom}(C_i, k^*)$-orbits of isomorphism classes of simple $k_*\hat{G}^i$-modules which remain simple restricted to $\hat{K}_i$, and the set of $\mathrm{Hom}(C, k^*)$-orbits of iso-morphism classes of the simple $k_*\hat{G}$-modules which remain simple restricted to $\hat{K}$.

But, it follows from [10, 15.23.2] that these sets of isomorphism classes respectively label $\mathcal{RG}_k(C_i)$- and $\mathcal{RG}_k(C)$-bases of $\mathcal{R}_{\hat{K}^i}\mathcal{G}_k(\hat{G}^i)$ and $\mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G})$; this implies the bijectivity of homomorphism 4.21.6 above, proving our claim. Moreover, it is easily checked that M_i is associated with the block b_i if and only if $\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}(M_i)$ is associated with the block b; hence, isomorphism 4.21.6 induces the $\mathcal{O}$-module isomorphism 4.21.1.

On the other hand, since any k^*-automorphism $\hat{\sigma}$ of $\hat{G}$ stabilizes the family $\{\hat{H}_j\}_{j\in I}$, if $\hat{\sigma}$ stabilizes b then it also stabilizes both families $\{\hat{K}_j\}_{j\in I}$ and $\{\hat{G}^j\}_{j\in I}$, and therefore, according to Lemma 4.10, $\hat{\sigma}$ can be lifted to an automorphism $\hat{\hat{\sigma}}$ of $\hat{\hat{G}}$; moreover, since $\hat{G}$ acts transitively on I, up to a modification of $\hat{\sigma}$ by an inner automorphism of $\hat{G}$, we may assume that $\hat{\sigma}$ fixes i and then $\hat{\hat{\sigma}}$ determines a k^*-automorphism $\hat{\sigma}_i$ of $\hat{G}^i$; in this case, assuming that M_i is associated with the block b_i, it is quite clear that

$$\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}\big(\mathrm{Res}_{\hat{\sigma}_i}(M_i)\big) \cong \mathrm{Res}_{\hat{\hat{\sigma}}}\big(\mathrm{Ten}_{\hat{\hat{G}}_i}^{\hat{\hat{G}}}(M_i)\big) \qquad 4.21.10$$

and therefore, since homomorphism 4.20.1 maps the class of $\hat{\sigma}_i$ on the class of $\hat{\sigma}$, homomorphism 4.21.6 is actually an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module homomor-phism. We are done.

4.22. Now, always choosing an element i in I and setting $\mathcal{F}^i = \mathcal{F}_{(b_i, \hat{G}^i)}$, we have an analogous result on the relationship between $\mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$ and $\mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}})$; here we need the alternative definition 2.12.3.

Theorem 4.23. *With the notation above, assume that C acts transitively on I and choose an element i of I. Then, there is an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism*

$$\mathcal{RG}_k(C) \otimes_{\mathcal{RG}_k(C_i)} \mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}}) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \qquad 4.23.1.$$

Proof: Let us recall our notation in [10, 15.33]; let $\mathfrak{r}:\Delta_n \to \mathcal{F}^{\mathrm{sc}}$ be a $\mathcal{F}^{\mathrm{sc}}$-chain (cf. 2.12); since $\mathfrak{r}(n)$ is also $\mathcal{K}$-selfcentralizing [10, Lemma 15.16] and we identify $\mathcal{F}(\mathfrak{r})$ with the stabilizer in $\mathcal{F}\big(\mathfrak{r}(n)\big)$ of the images of $\mathfrak{r}(\ell)$ in $\mathfrak{r}(n)$ for any $\ell \in \Delta_n$, it makes sense to consider $\mathcal{K}(\mathfrak{r}) = \mathcal{K}\big(\mathfrak{r}(n)\big) \cap \mathcal{F}(\mathfrak{r})$ which is a normal subgroup of $\mathcal{F}(\mathfrak{r})$; then, we denote by $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$ the full subcategory of $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{sc}})$ over the set of $\mathcal{F}^{\mathrm{sc}}$-chains $\mathfrak{r}:\Delta_n \to \mathcal{F}^{\mathrm{sc}}$ such that

$$\mathcal{F}(\mathfrak{r})/\mathcal{K}(\mathfrak{r}) \cong C \qquad\qquad 4.23.2.$$

More explicitly, by the very definition of $\mathcal{F} = {}^C\mathcal{K}$, we have

$$\mathcal{F}(Q)/\mathcal{K}(Q) \cong C \qquad\qquad 4.23.3;$$

choosing a lifting $\sigma \in \mathcal{F}(Q)$ of a generator of C, we have a *Frobenius functor* $\mathfrak{f}_\sigma : \mathcal{F} \to \mathcal{F}$ [10, 12.1] and therefore we get a new $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{f}_\sigma \circ \mathfrak{r}$; then, isomorphism 4.23.2 is equivalent to the existence of a *natural* isomorphism $\nu : \mathfrak{r} \cong \mathfrak{f}_\sigma \circ \mathfrak{r}$ formed by $\mathcal{K}$-isomorphisms.

Mutatis mutandis, we also consider the corresponding full subcategory $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$ of $\mathfrak{ch}^*\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$. Then, it follows from [10, 15.36] that we have *contravariant* functors

$$\mathcal{R}_{\check{\mathcal{K}}(\bullet)}\mathcal{G}_k\big(\hat{\mathcal{F}}(\bullet)\big) : \mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}}) \longrightarrow \mathcal{O}\text{-}\mathfrak{mod}$$
$$\mathcal{R}_{\check{\mathcal{K}}^i(\bullet)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\bullet)\big) : \mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big) \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \qquad 4.23.4$$

respectively mapping any $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-object $(\mathfrak{r}, \Delta_n)$ on $\mathcal{R}_{\check{\mathcal{K}}(\mathfrak{r})}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r})\big)$ and any $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-object $(\mathfrak{r}_i, \Delta_n)$ on $\mathcal{R}_{\check{\mathcal{K}}^i(\mathfrak{r}_i)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\mathfrak{r}_i)\big)$, and from [10, Proposition 15.37] that we still have

$$\mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \cong \varprojlim \mathcal{R}_{\check{\mathcal{K}}(\bullet)}\mathcal{G}_k\big(\hat{\mathcal{F}}(\bullet)\big)$$
$$\mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}}) \cong \varprojlim \mathcal{R}_{\check{\mathcal{K}}^i(\bullet)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\bullet)\big) \qquad 4.23.5.$$

Explicitly, we have

$$\mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}) \subset \prod_{\mathfrak{r}} \mathcal{R}_{\check{\mathcal{K}}(\mathfrak{r})}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r})\big) \qquad 4.23.6,$$

where $(\mathfrak{r}, \Delta_m)$ runs over the set of $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-objects, and the left member coincides with the set of $(X_{\mathfrak{r}})_{\mathfrak{r}}$, where $X_{\mathfrak{r}} \in \mathcal{R}_{\check{\mathcal{K}}(\mathfrak{r})}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r})\big)$, which are "stable" by $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-isomorphisms and, for any $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-object $(\mathfrak{q}, \Delta_n)$ such that $\mathfrak{q} = \mathfrak{r} \circ \iota$ for some injective order-preserving map $\iota:\Delta_n \to \Delta_m$, the corresponding restriction map sends $X_{\mathfrak{q}}$ to $X_{\mathfrak{r}}$; in particular, it is clear that we can restrict ourselves to the $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-objects such that we have $\mathfrak{r}(\ell-1) \subset \mathfrak{r}(\ell)$ and $\mathfrak{r}(\ell-1 \bullet \ell)$ is the inclusion map for any $1 \le \ell \le m$.

Moreover, for any $\ell \in \Delta_m$, let us denote by $\mathfrak{r}_j(\ell)$ the image of $\mathfrak{r}(\ell)$ in Q_j and consider the $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{r}^* : \Delta_m \to \mathcal{F}^{\mathrm{sc}}$ mapping $\ell \in \Delta_m$ on $\prod_{j \in I} \mathfrak{r}_j(\ell)$ and the Δ_m-morphisms on the corresponding inclusions; since $\mathcal{K}$ is *normal* in $\mathcal{F}$ [10, 12.6], it follows from [10, Proposition 12.8] that any $\mathcal{F}$-automorphism of $\mathfrak{r}(m)$ induces an $\mathcal{F}$-automorphism of $\mathfrak{r}^*(m)$; in particular, we get a group homomorphism $\mathcal{F}(\mathfrak{r}) \to \mathcal{F}(\mathfrak{r}^*)$ and therefore $\mathfrak{r}^*$ also fulfills condition 4.23.2; furthermore, our functor $\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}}$ lifts this homomorphism to a k^*-group homomorphism $\hat{\mathcal{F}}(\mathfrak{r}) \to \hat{\mathcal{F}}(\mathfrak{r}^*)$.

Consequently, considering the corresponding full subcategory, it follows from [10, Proposition A4.7] that, in the direct product in 4.23.6 above, we can restrict ourselves to the $\mathcal{F}^{\mathrm{sc}}$-chains $\mathfrak{r}$ such that we have $\mathfrak{r}^* = \mathfrak{r}$; more explicitly, we may assume that, for any $\ell \in \Delta_m$, we have

$$\mathfrak{r}(\ell) = \prod_{j \in I} \mathfrak{r}_j(\ell) \qquad\qquad 4.23.7$$

where, for any $j \in I$, $\mathfrak{r}_j : \Delta_m \to (\mathcal{F}^j)^{\mathrm{sc}}$ is a $(\mathcal{F}^j)^{\mathrm{sc}}$-chain, fulfilling the corresponding condition 4.23.2, such that we have $\mathfrak{r}_j(\ell-1) \subset \mathfrak{r}_j(\ell)$ and $\mathfrak{r}_j(\ell-1 \bullet \ell)$ is the inclusion map for any $1 \le \ell \le m$.

In this case, it is quite clear that

$$\hat{\mathcal{K}}(\mathfrak{r}) \cong \hat{\prod}_{j \in I} \hat{\mathcal{K}}^j(\mathfrak{r}_j) \qquad\qquad 4.23.8$$

and, arguing as in 4.19 and 4.20 above, it follows from Proposition 4.21 above that we have a *canonical* $\mathcal{R}\mathcal{G}_k(C)$-isomorphism

$$\rho_{\mathfrak{r}} : \mathcal{R}_{\hat{\mathcal{K}}(\mathfrak{r})}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r})\big) \cong \mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\hat{\mathcal{K}}^i(\mathfrak{r}_i)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\mathfrak{r}_i)\big) \qquad 4.23.9$$

Mutatis mutandis, we have

$$\mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}}) \subset \prod_{\mathfrak{r}_i} \mathcal{R}_{\hat{\mathcal{K}}^i(\mathfrak{r}_i)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\mathfrak{r}_i)\big) \qquad\qquad 4.23.10$$

where $(\mathfrak{r}_i, \Delta_m)$ runs over the set of $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-objects; once again, we can restrict ourselves to the $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-objects such that $\mathfrak{r}_i(\ell-1) \subset \mathfrak{r}_i(\ell)$ and $\mathfrak{r}_i(\ell-1 \bullet \ell)$ is the inclusion map for any $1 \le \ell \le m$. In particular, the extension

$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}}) \qquad\qquad 4.23.11$$

coincides with the set of $(X_{\mathfrak{r}_i})_{\mathfrak{r}_i}$, where $(\mathfrak{r}_i, \Delta_m)$ runs over the set of such $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-objects and $X_{\mathfrak{r}_i}$ belongs to the extension

$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\hat{\mathcal{K}}^i(\mathfrak{r}_i)}\mathcal{G}_k\big(\hat{\mathcal{F}}^i(\mathfrak{r}_i)\big) \qquad\qquad 4.23.12,$$

which are "stable" by $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-isomorphisms and, moreover, for such a $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-object $(\mathfrak{q}_i, \Delta_n)$ fulfilling $\mathfrak{q}_i = \mathfrak{r}_i \circ \iota$ for some injective order-preserving map $\iota : \Delta_n \to \Delta_m$, the corresponding restriction map sends $X_{\mathfrak{q}_i}$ to $X_{\mathfrak{r}_i}$.

On the other hand, for any subgroup R_i of $Q_i \subset Q$, we consider the subgroup of Q

$$R_i^* = \prod_{0 \leq t < |I|} \sigma^t(R_i) \qquad\qquad 4.23.13$$

and, more generally, for any $(\mathcal{F}^i)^{\mathrm{sc}}$-chain $\mathfrak{r}_i$ defined by inclusion maps, we denote by $\mathfrak{r}_i^*$ the corresponding $\mathcal{F}^{\mathrm{sc}}$-chain. Further, if $\mathfrak{r}_i$ is a $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-object then $\mathfrak{r}_i^*$ is a $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-object; indeed, the $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{f}_\sigma \circ \mathfrak{r}_i^*$ maps $\ell \in \Delta_m$ on

$$\sigma^{|I|}\big(\mathfrak{r}_i(\ell)\big) \times \prod_{1 \leq t < |I|} \sigma^t\big(\mathfrak{r}_i(\ell)\big) \qquad\qquad 4.23.14;$$

but, we are assuming that there is a *natural* isomorphism $\mathfrak{r}_i \cong \mathfrak{f}_{\sigma^{|I|}} \circ \mathfrak{r}_i$ formed by $\mathcal{K}^i$-isomorphisms; hence, we have a *natural* isomorphism $\mathfrak{r}_i^* \cong \mathfrak{f}_\sigma \circ \mathfrak{r}_i^*$ formed by $\mathcal{K}$-isomorphisms.

Moreover, it is quite clear that any $\mathfrak{ch}^*\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-isomorphism $\mathfrak{r}_i \cong \mathfrak{r}_i'$ can be lifted to a $\mathfrak{ch}^*(\mathcal{F}^{\mathrm{sc}})$-isomorphism $\mathfrak{r}_i^* \cong \mathfrak{r}_i'^*$, and that $\mathfrak{q}_i = \mathfrak{r}_i \circ \iota$ forces $\mathfrak{q}_i^* = \mathfrak{r}_i^* \circ \iota$. Consequently, if $(X_{\mathfrak{r}})_{\mathfrak{r}}$ is an element of $\mathcal{R}_\mathcal{K}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$ then $\big(\rho_{\mathfrak{r}_i^*}(X_{\mathfrak{r}_i^*})\big)_{\mathfrak{r}_i}$ is clearly an element of $\mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathfrak{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}})$.

Conversely, for any $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-object $(\mathfrak{r}, \Delta_m)$ such that $\mathfrak{r}$ is defined by inclusion maps and fulfills equality 4.23.7, it is quite clear that the corresponding $(\mathcal{F}^i)^{\mathrm{sc}}$-chain $\mathfrak{r}_i$ is also defined by inclusion maps and that $(\mathfrak{r}_i, \Delta_m)$ is a $\mathfrak{ch}^*_{C_i}\big((\mathcal{F}^i)^{\mathrm{sc}}\big)$-object; then, from isomorphism 4.23.2 it is easy to check that $\mathfrak{r}_i^*$ is *naturally* isomorphic to $\mathfrak{r}$ *via* an isomorphism inducing the identity on $\mathfrak{r}_i$. Finally, for any element $(X_{\mathfrak{r}_i})_{\mathfrak{r}_i}$ of the extension 4.23.12 and any $\mathfrak{ch}^*_C(\mathcal{F}^{\mathrm{sc}})$-object $(\mathfrak{r}, \Delta_m)$ as above, we can define $X_{\mathfrak{r}}$ as the image of $(\rho_{\mathfrak{r}_i^*})^{-1}(X_{\mathfrak{r}_i})$ by the $\mathcal{R}\mathcal{G}_k(C)$-module isomorphism

$$\mathcal{R}_{\hat{\mathcal{K}}(\mathfrak{r}_i^*)}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r}_i^*)\big) \cong \mathcal{R}_{\hat{\mathcal{K}}(\mathfrak{r})}\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{r})\big) \qquad\qquad 4.23.15$$

determined by a *natural* isomorphism $\mathfrak{r}_i^* \cong \mathfrak{r}$ inducing the identity on $\mathfrak{r}_i$; it is easily checked that this definition does not depend on the choice of the *natural* isomorphism $\mathfrak{r}_i^* \cong \mathfrak{r}$ inducing the identity on $\mathfrak{r}_i$, and that the element $(X_{\mathfrak{r}})_{\mathfrak{r}}$ of the corresponding direct product actually belongs to $\mathcal{R}_\mathcal{K}\mathcal{G}_k(\mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{nc}}})$ (cf. 4.23.6). It is clear that both correspondences are inverse of each other and therefore they define the announced isomorphism 4.23.1.

Corollary 4.24. *With the notation above, assume that C acts transitively on I. Then, if* (Q) *holds for $(b_i, {}^{D_i}\hat{K}^i)$ for any $i \in I$ and any subgroup D_i of C_i, it holds for $(b, \hat{G})$.*

Proof: It follows from Propositions 4.21 and 3.23 that, choosing $i \in I$, we have $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphisms

$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\hat{K}^i}\mathcal{G}_k(\hat{G}^i, b_i) \cong \mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G}, b)$$
$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(C_i)} \mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathrm{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}}) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}})$$

4.24.1.

On the other hand, assume that we have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}^i)_{b_i}$-module isomorphism

$$\mathcal{G}_k(\hat{G}^i, b_i) \cong \mathcal{G}_k(\mathcal{F}^i, \widehat{\mathrm{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}})$$

4.24.2;

then, since the restriction induces compatible $\mathcal{G}_k(C_i)$-module structures on both members of this isomorphism [10, 15.21 and 15.33], it follows from [10, 15.23.2 and 15.37.1] that we still have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G}^i)_{b_i}$-module isomorphism

$$\mathcal{R}_{\hat{K}_i}\mathcal{G}_k(\hat{G}^i, b_i) \cong \mathcal{R}_{\mathcal{K}^i}\mathcal{G}_k(\mathcal{F}^i, \widehat{\mathrm{aut}}_{(\mathcal{F}^i)^{\mathrm{nc}}})$$

4.24.3.

Thus, from isomorphisms 4.24.1 above, we get an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(\hat{G}, b) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(\mathcal{F}, \widehat{\mathrm{aut}}_{(\mathcal{F})^{\mathrm{nc}}})$$

4.24.4.

Consequently, according to our hypothesis and possibly applying Proposition 4.18 above and [10, 15.23.2 and 15.37.1], for any subgroup D of C we have an $\mathcal{O}\mathrm{Out}_{k^*}(^D\hat{K})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k(^D\hat{K}, b) \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k(^D\mathcal{K}, \widehat{\mathrm{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}})$$

4.24.5;

but, since $\mathrm{Aut}_{k^*}(\hat{G})_b$ stabilizes $\hat{K}$, we have evident group homomorphisms

$$C \longrightarrow \mathrm{Out}_{k^*}(^D\hat{K})_b \longleftarrow \mathrm{Aut}_{k^*}(\hat{G})_b$$

4.24.6

and it is clear that the image of $\mathrm{Aut}_{k^*}(\hat{G})_b$ contains and normalizes the image of C; hence, we still have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{R}_{\hat{K}}\mathcal{G}_k\big(^D\hat{K}, b\big)^C \cong \mathcal{R}_{\mathcal{K}}\mathcal{G}_k\big(^D\mathcal{K}, \widehat{\mathrm{aut}}_{(^D\mathcal{K})^{\mathrm{nc}}}\big)^C$$

4.24.7.

Then, it follows from [10, 15.23.4 and 15.38.1] that the direct sum of isomorphisms 4.24.6 when D runs over the set of subgroups of C supplies an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism $\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}, \widehat{\mathrm{aut}}_{\mathcal{F}^{\mathrm{nc}}})$. We are done.

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Abstract

We show that the refinement of Alperin's Conjecture proposed in [10, Ch. 16] can be proved by checking that this refinement holds on any central k^*-extension of a finite group H containing a normal simple group S with trivial centralizer in H and p'-cyclic quotient H/S. This paper improves our result in [10, Theorem 16.45] and repairs some bad arguments there.